Lead Time and Ordering Cost Reductions are Interdependent in Inventory Model with Backorder Price Discount

Yu-Jen Lin[1]

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Abstract

The stochastic inventory model analyzed in this paper explore the problem that the lead time and ordering cost reductions are interdependent in the continuous review inventory model with backorder discount. The objective of this paper is to minimize the total related cost by simultaneously optimizing the order quantity, lead time and backorder price discount. Moreover, the lead time demand is assumed to be normally distributed. A procedure of finding the optimal solution is developed. Furthermore, the sensitivity analysis is included and two numerical examples are given to illustrate the results.

Keywords**:** inventory, lead time, backorder price discount.

[1]Holistic Education Center, St. John's University

1. Introduction

In most of early literature dealing with inventory problems, either using deterministic or probabilistic models, lead time is viewed as a prescribed constant or a stochastic variable, which therefore, is not subject to control (see, e.g., Johnson and Montgomery (1974), Naddor (1966), Silver and Peterson (1985)). However, this may not be realistic. In some practical cases, lead time can be shortened at an added crashing cost; in other words, it is controllable. By shortening the lead time, we can lower the safety stock, reduce the stockout loss and improve the service level to the customer so as to increase the competitive edge in business.

On the other hand, in the real market, at the manufacturing level, a supply stockout of an item usually results in a backorder, while at the retail level, a stockout may result in a lost sale because the customer will refuse the backorder case, and go to elsewhere to make the purchase. Therefore, when the inventory system unsatisfied demands occur during the stockout period, how to provide the price discount from supplier such that the customers willing to wait for the backorder, which is very important. The price discount is a potential factor that may motivate the customers' desire for backorders. In general, provided that a supplier could offer a price discount on the stockout item by negotiation to secure more backorders, it may make the customers more willing to wait for the desired items. In other words, the bigger the discount, the bigger the advantage to the customers, and hence, a larger number of backorder ratio may result. This phenomenon reveals that, as unsatisfied demands occur during the stockout period, how to find an optimal backorder ratio through controlling a price discount from a supplier to minimize the relevant inventory total cost is a decision-making problem worth discussing.

In recent years, several authors have presented various inventory models with lead time reduction. Initially, Liao and Shyu (1991) presented an inventory model in which lead time is a unique decision variable and the order quantity is predetermined. Ben-Daya and Raouf (1994) extended Liao and Shyu's (1991) model by allowing both the lead time and the order quantity as decision variables. Ouyang et al. (1996) generalized Ben-Daya and Raouf's (1994) model by allowing shortages with partial backorders. Pan and Hsiao (2001) revised Ouyang et al.'s (1996) model to consider the backorder discount as one of the decision variables, while Chuang et al.(2004) extended Pan and Hsiao (2001) model develop a minimax distribution free procedure for inventory model with backorder discount and variable lead time.

It is noticed that the above papers (Ben-Daya and Raouf (1994), Chuang et al.(2004), Liao and Shyu's (1991), Ouyang et al.'s (1996), Pan and Hsiao (2001)) are all focusing on the continuous review inventory model to derive the benefits from lead time reduction, and the ordering cost is treated as a fixed constant. In a recent paper, Ouyang et al. (1999) proposed continuous review inventory model to study the effects of lead time and ordering cost reductions. We note that the lead time and ordering cost reductions in Ouyang et al.' s models (1999) are assumed to act independently; however, this is only one of the possible cases. In practice, the lead time and ordering cost reductions may be related closely; the reduction of lead time may accompany the reduction of ordering cost, and vice versa. For example, according to Silver and Peterson (1985), the implementation of electronic data interchange (EDI) may reduce the lead time and ordering cost simultaneously. Therefore, it is more reasonable to assume that ordering cost reduction vary according to different lead times. And then, their functional relationship may be as linear, logarithmic, exponential and the likes.

In this paper is to investigate the effect of lead time reduction on a continuous review inventory model includes the controllable backorder price discount and the reduction of lead time accompanies a decrease of ordering cost. We assume that the lead time demand is normal distribution, and seek to minimize the total related cost by optimizing the order quantity, backorder discount and lead time simultaneously.

The rest of this paper is organized as follows. In the next section, the notation and assumptions are presented. The model that the lead time demand has perfect information is formulated in Section 3. Two numerical examples are provided to illustrate the proposed model and sensitivity analysis of the optimal solutions with respect to parameters are also indicated in Section 4, and Section 5 is a conclusions.

2. Notation and assumptions

The notation used here are:

D = average demand per year

A_0 = original ordering cost

A = ordering cost per order, $0 < A \le A_0$

h = inventory holding cost per unit per year

Q = order quantity (a decision variable)

r = reorder point

b = fraction of the shortage that will be backordered, i.e., backorder ratio, $0 \le b < 1$

b_0 = upper bound of the backorder ratio

p_x = backorder price discount offered by the supplier per unit (a decision variable)

p_0 = marginal profit (i.e. cost of lost demand) per unit

L = length of lead time (a decision variable)

X = lead time demand

$f_X(x)$ = the probability density function (*p.d.f.) of* X with finite mean DL and standard deviation $s\sqrt{L}$, where s denotes the standard deviation of the demand per unit time

$E(\cdot)$ = mathematical expectation

x^+ = maximum value of x and 0, i.e., $x^+ = Max\{x,0\}$

The assumptions of the model are:

1. The reorder point r = expected demand during lead time + safety stock (SS), and SS = k ×(standard deviation of lead time demand), i.e., $r = DL + ks\sqrt{L}$ where k is the safety factor and satisfies $P(X > r) = q$, q is given to represent the allowable stockout probability during L.
2. Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point r.
3. The lead time L consists of n mutually independent components. The i-th component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Further, for convenience, we rearrange c_i such that $c_1 \le c_2 \le \ldots \le c_n$. Then, it is clear that the reduction of lead time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.
4. We let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components 1,2,…, i crashed to their minimum duration, then L_i can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j)$, i =1,2,…, n; and the lead time crashing cost, $C(L)$, per cycle for a given $L \in [L_i, L_{i-1}]$ is given by $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.
5. The reduction of lead time L accompanies a decrease of ordering cost A, and A is a strictly concave function of L, i.e., $A'(L) > 0$ and $A''(L) \le 0$.
6. The supplier makes decisions in order to obtain profits. Therefore, if the price discount, p_x, is greater than the marginal profit, p_0, then the supplier may decide against offering the price discount.
7. During the stockout period, the backorder ratio, b, is variable and is in proportion to the price discount, p_x, offered by the supplier per unit. The backorder rate is defined as $b = b_0 p_x / p_0$, where $0 \le b_0 < 1$ and $0 \le p_x \le p_0$.

3. Models formulation

In a recent study, Pan and Hsiao (2001) and Chuang et al. (2004) consider the inventory model with backorder discount and variable lead time. They assumed that the ordering cost is treated as a fixed constant and independent of lead time. In this study, we will closely follow Pan and Hsiao (2001) and Chuang et al. (2004), which the lead time demand follows the normal distribution for the backorder discount on the continuous review inventory model involving the ordering cost dependent on lead time. Specifically, by assumptions 1-5, the total expected annual cost, which is composed of ordering cost, inventory holding cost, stockout cost and lead time crashing cost, is expressed by

$$EAC(Q,b,L) = \frac{A(L)D}{Q} + h\left[\frac{Q}{2} + r - DL + (1-b)E(X-r)^{+}\right] + \frac{D}{Q}[p_x b + p_0(1-b)]E(X-r)^{+} + \frac{D}{Q}C(L), \qquad (1)$$

where $E(X-r)^{+}$ is the expected demand shortage at the end of cycle. Further, by assumption 7, during the stockout period, the backorder ratio, b, is variable and is proportion to the price discount offered by the supplier per unit, p_x, that is, $b = b_0 p_x / p_0$. Therefore, the backorder price discount offered by the supplier per unit, p_x, can be treated as a decision variable instead of the backorder ratio, b. That is, the objective of cost function (1) is to minimize the following total expected annual cost.

$$EAC(Q,p_x,L) = \frac{A(L)D}{Q} + h\left[\frac{Q}{2} + r - DL + \left(1 - \frac{b_0 p_x}{p_0}\right)E(X-r)^{+}\right] + \frac{D}{Q}\left(\frac{b_0 p_x^{\,2}}{p_0} + p_0 - b_0 p_x\right)E(X-r)^{+} + \frac{D}{Q}C(L). \qquad (2)$$

Moreover, when the lead time demand X follows a normal distribution has a $p.d.f.\ f_X(x)$ with finite mean DL and standard deviation $s\sqrt{L}$, by using $r = DL + ks\sqrt{L}$, the expected demand shortage at the end of the cycle $E(X-r)^{+} = \int_r^{\infty}(x-r)f_X(x)dx = s\sqrt{L}\,y(k)$, where $y(k) \equiv f(k) - k[1-\Phi(k)] > 0$, $f(k)$ and $\Phi(k)$ denote the standard normal probability density function (*p.d.f.*) and distribution function (*d.f.*), respectively. Thus, the total expected annual cost function (2) can be transformed into following formulation.

$$EAC(Q,p_x,L)=\frac{A(L)D}{Q}+h\left[\frac{Q}{2}+ks\sqrt{L}\right]$$
$$+\left[h\left(1-\frac{b_0p_x}{p_0}\right)+\frac{D}{Q}G(p_x)\right]s\sqrt{L}\,y(k)+\frac{D}{Q}C(L)\,, \tag{3}$$

where $G(p_x)=p_0-b_0p_x+\frac{b_0p_x^{\ 2}}{p_0}>0$ (because $\frac{p_0}{p_x}>b_0\left(1-\frac{p_x}{p_0}\right)>0$).

In order to determine the optimal values of Q, p_x and L, respectively, such that $EAC(Q,p_x,L)$ in (3) is minimized. Taking the first partial derivatives of $EAC(Q,p_x,L)$ with respect to Q, p_x and $L\in[L_i,L_{i-1}]$, respectively. We obtain:

$$\frac{\partial EAC(Q,p_x,L)}{\partial Q}=-\frac{A(L)D}{Q^2}+\frac{h}{2}-\frac{DG(p_x)s\sqrt{L}\,y(k)}{Q^2}-\frac{D}{Q^2}C(L)\,, \tag{4}$$

$$\frac{\partial EAC(Q,p_x,L)}{\partial p_x}=\left[\frac{D}{Q}\left(\frac{2b_0p_x}{p_0}-b_0\right)-\frac{hb_0}{p_0}\right]s\sqrt{L}\,y(k)\,, \tag{5}$$

and

$$\frac{\partial EAC(Q,p_x,L)}{\partial L}=\frac{A'(L)D}{Q}+\frac{hks}{2\sqrt{L}}+\frac{1}{2\sqrt{L}}\left[h\left(1-\frac{b_0p_x}{p_0}\right)+\frac{D}{Q}G(p_x)\right]s\,y(k)-\frac{Dc_i}{Q}\,. \tag{6}$$

By examining the second order sufficient conditions (SOSC), it can be easily verified that $EAC(Q,p_x,L)$ is not a convex function of (Q,p_x,L). However, for fixed Q and p_x, $EAC(Q,p_x,L)$ is concave in $L\in[L_i,L_{i-1}]$, because

$$\frac{\partial^2 EAC(Q,p_x,L)}{\partial L^2}=\frac{A''(L)D}{Q}-\frac{1}{4}hks\,L^{-\frac{3}{2}}-\frac{1}{4}\left[h\left(1-\frac{b_0p_x}{p_0}\right)+\frac{DG(p_x)}{Q}\right]s\,L^{-\frac{3}{2}}\,y(k)$$

<0. Therefore, for fixed Q and p_x, the minimum total expected annual cost will occur at the end points of the interval $[L_i,L_{i-1}]$. On the other hand, for a given value of $L\in[L_i,L_{i-1}]$, $EAC(Q,p_x,L)$ is convex in both Q and p_x (see Appendix for a detail proof). Thus, for fixed $L\in[L_i,L_{i-1}]$, the minimum value of $EAC(Q,p_x,L)$ will occur at the point (Q,p_x), which satisfies $\dfrac{\partial EAC(Q,p_x,L)}{\partial Q}=0$ and $\dfrac{\partial EAC(Q,p_x,L)}{\partial p_x}=0$ simultaneously. By setting equations (4) and (5) equal to zero, we obtain

$$Q=\left[\frac{A(L)D+DG(p_x)s\sqrt{L}\,y(k)+DC(L)}{h/2}\right]^{\frac{1}{2}}, \tag{7}$$

and

$$p_x=\frac{hQ}{2D}+\frac{p_0}{2}, \tag{8}$$

respectively.

Substituting equation (8) into equation (7) and simplify, we get

$$Q=\left\{\frac{2D\left[A(L)+\dfrac{p_0(4-b_0)}{4}s\sqrt{L}\,y(k)+C(L)\right]}{h\left[1-\dfrac{b_0h}{2p_0D}s\sqrt{L}\,y(k)\right]}\right\}^{\frac{1}{2}}. \tag{9}$$

It note that, from (7) and $G(p_x)>0$, the order quantity, Q, is greater than zero.

Theoretically, for given $L\in[L_i,L_{i-1}]$, and k (which depends on the allowable stockout probability q and the *p.d.f.* $f_X(x)$) from equations (8) and (9), we can obtain the optimal solution (Q,p_x), such that the total expected annual cost has minimum values. Therefore, we develop the following algorithm to find the optimal solution for the order quantity, price discount and lead time.

Algorithm

Step1. For each L_i, $i=0,1,2,\cdots,n$ and a given q (and hence, the value of k can be found directly from the standard normal distribution table), compute Q_i from equation (9) and then determine p_{x_i} from equation (8). And compare p_{x_i} and p_0.

(i) If $p_{x_i}\le p_0$, p_{x_i} is feasible, then go to Step2.

(ii) If $p_{x_i}>p_0$, p_{x_i} is not feasible. Set $p_{x_i}=p_0$ and calculate the corresponding value of Q_i from equation (7), then go to Step2.

Step2. For each (Q_i, p_{x_i}, L_i), compute the corresponding total expected annual cost

$EAC(Q_i, p_{x_i}, L_i)$, i =0, 1, 2, …, n.

Step3. Find $\underset{i=0,1,2,\ldots,n}{Min}\ EAC(Q_i, p_{x_i}, L_i)$. If $EAC(Q^*, p_x^{\ *}, L^*) = \underset{i=0,1,2,\ldots,n}{Min}\ EAC(Q_i, p_{x_i}, L_i)$,

then $(Q^*, p_x^{\ *}, L^*)$ is the optimal solution.

Once we obtain the $(Q^*, p_x^{\ *}, L^*)$, the optimal reorder point is $r^* = DL^* + ks\sqrt{L^*}$, the optimal backorder ratio is $b^* = b_0 p_x^{\ *} / p_0$ and the optimal ordering cost $A^* = A(L^*)$ follows.

4. Numerical examples

The numerical examples given below illustrate the above solution procedure. We consider the continuous review inventory system with the following data used in (Pan and Hsiao (2001), Chuang et al. (2004)): D = 600 units per year, A = \$200 per order, h = \$20 per unit per year, \$150 per unit, 7 units per week. Besides, we assume that the lead time has three components with data shown in Table 1.

Table 1 Lead time data

Lead time Component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Example 1. We assume that lead time and ordering cost reductions act dependently with the following relationship (see, Chen et al.(2001)): $\dfrac{A_0 - A}{A_0} = \dfrac{1}{l}\left(\dfrac{L_0 - L}{L_0}\right)$, which implies $A(L) = a + bL$, where $l > 0$, $a = \left(1 - \dfrac{1}{l}\right)A_0$ *and* $b = \dfrac{A_0}{l\,L_0}$. We attempt to solve the cases when the upper bound of the backorder ratio b_0 = 0.8, and the scaling parameter

l = 0.75, 1.00, 1.25, 2.50, 5.00 and q = 0.2 (in this situation, the value of safety factor k can be found directly from the standard normal distribution table, and is 0.845). Applying the Algorithm procedure yields the results as tabulated in Table 2. From this table, the optimal inventory policy can easily be found by comparing $EAC(Q_i, p_{x_i}, L_i)$, for $i = 0, 1, 2, 3$, and thus we summarize these in Table 3. Furthermore, in order to see the effect of lead time reduction with interaction of ordering cost, we list the results of fixed ordering cost model setting $A = \$200$ per order (i.e., take $l = \infty$) in the same table. From the results shown in Table 3, it reveals that as the value of l decreases, the larger savings of total expected annual cost are obtained (comparing the result with fixed ordering cost model). And it is interesting to observe that decreasing the value l will result in a decrease in the total expected annual cost, the order quantity, the backorder price discount.

Table 2 Solution procedures of Example 1 (L_i *in weeks*)

l	i	L_i	$C(L_i)$	$A(L_i)$	Q_i	p_{x_i}	r_i	$EAC(Q_i, p_{x_i}, L_i)$
0.75	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	133.33	148.45	77.47	83.71	3,281.19
	2	4	22.4	66.67	128.57	77.14	57.98	2,826.38
	3	3	57.4	33.33	123.01	77.05	44.86	2,681.03 *
1.00	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	150.00	151.78	77.52	83.71	3,347.80
	2	4	22.4	100.00	136.13	77.26	57.98	2,977.49
	3	3	57.4	75.00	132.78	77.21	44.86	2,876.49 *
1.25	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	160.00	153.74	77.56	83.71	3,387.08
	2	4	22.4	120.00	140.47	77.34	57.98	3,064.26
	3	3	57.4	100.00	138.32	77.30	44.86	2,987.15 *
2.50	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	180.00	157.60	77.62	83.71	3,464.16
	2	4	22.4	160.00	148.76	77.47	57.98	3,230.21
	3	3	57.4	150.00	148.77	77.47	44.86	3,196.14 *
5.00	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	190.00	159.49	77.65	83.71	3,502.00
	2	4	22.4	180.00	152.74	77.54	57.98	3,309.81
	3	3	57.4	175.00	153.73	77.56	44.86	3,295.31 *

Besides, we further examine the effects of changes in the system parameters h, D, A and p_0 on the optimal order quantity Q^*, optimal backorder discount $p_x{}^*$ and minimum total expected annual cost $EAC(Q^*, p_x{}^*, L^*)$ in Example 1. When the upper bound of the backorder ratio $b_0 = 0.8$ and the scaling parameter $l = 1$, a sensitivity analysis is performed by changing each of the parameters by +50%, +25%, –25% and –50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 4.

Table 3 Summary of the optimal solution of Example 1 (L^* *in weeks*)

l	L^*	A^*	Q^*	$p_x{}^*$	r^*	$EAC(Q^*, p_x{}^*, L^*)$	*Saving*(%)
0.75	3	33.33	123.01	77.05	44.86	2,681.03	20.85
1.00	3	75.00	132.78	77.21	44.86	2,876.49	15.08
1.25	3	100.00	138.32	77.30	44.86	2,987.15	11.82
2.50	3	150.00	148.77	77.47	44.86	3,196.14	5.65
5.00	3	175.00	153.73	77.56	44.86	3,295.31	2.72
∞	4	200.00	156.62	77.61	57.98	3,387.38	--

Note: Saving is based on the fixed ordering cost (i.e., $l = \infty$).

Table 4 Effects of changes in the parameters of the inventory model of Example 1

parameters	% change	% change in		
		Q^*	$p_x{}^*$	$EAC(Q^*, p_x{}^*, L^*)$
h	+50	–5.46	+1.19	+42.41
	+25	+3.55	+0.84	+29.09
	–25	+33.68	+0.00	–1.67
	–50	+63.72	–0.51	–20.58
D	+50	+41.79	–0.15	+38.58
	+25	+29.43	+0.10	+27.17
	–25	+0.26	+0.95	+0.23
	–50	–18.12	+1.82	–16.70
A	+50	+27.99	+0.80	+25.83
	+25	+22.03	+0.63	+20.34

	–25	+9.15	+0.25	+8.44
	–50	+2.10	+0.05	+1.94
p_0	+50	+27.08	+49.93	+25.00
	+25	+21.56	+24.90	+19.90
	–25	+9.68	–34.37	+8.93
	–50	+3.23	–48.47	+2.97

On the basis of the results of Table 4, the following observations can be made.

(i) p_x^* and $EAC(Q^*, p_x^*, L^*)$ increase while Q^* decreases with an increase in the value of the model parameter h. The obtained results show that Q^* and $EAC(Q^*, p_x^*, L^*)$ are highly sensitive whereas p_x^* is lowly sensitive to the changes in h.

(ii) Q^* and $EAC(Q^*, p_x^*, L^*)$ increase while p_x^* decreases with an increase in the value of the model parameter D. Moreover, Q^* and $EAC(Q^*, p_x^*, L^*)$ are highly sensitive whereas p_x^* is lowly sensitive to the changes in D.

(iii) Q^*, p_x^* and $EAC(Q^*, p_x^*, L^*)$ increase with an increase in the value of the model parameter A. Moreover, Q^* and $EAC(Q^*, p_x^*, L^*)$ are moderately sensitive whereas p_x^* is lowly sensitive to the changes in A.

(iv) Q^*, p_x^* and $EAC(Q^*, p_x^*, L^*)$ increase with an increase in the value of the parameter p_0. Moreover, Q^* and $EAC(Q^*, p_x^*, L^*)$ are moderately sensitive whereas p_x^* is highly sensitive to the changes in p_0.

Example 2. The data are the same as in Example 1, and we assume that the lead time and ordering cost reductions act dependently with the following relationship (see, Chen et al.(2001)): $\frac{A_0 - A}{A_0} = m \ln\left(\frac{L}{L_0}\right)$, which implies $A(L) = f + g \ln L$, where $m < 0$,

$f = A_0(1+m\ln L_0)$ and $g = -m A_0 > 0$. We solve the cases when the upper bound of the backorder ratio $b_0 = 0.8$, and the scaling parameter $m =$ -0.20, -0.50, -0.80, -1.00 and $q = 0.2$. Utilizing the similar procedure as Algorithm, we obtain the results as tabulated in Table 5. From this table, the optimal inventory policy can easily be found by comparing $EAC(Q_i, p_{x_i}, L_i)$, for $i = 0, 1, 2, 3$, and thus we summarize these in Table 6. Moreover, in order to observe the relationships between lead time and ordering cost, we list the results of fixed ordering cost model (i.e., take $m = 0$) in the same table. From the results shown in Table 5, we see that as the value of m decreases, the larger savings of total expected annual cost are obtained (comparing the result with fixed ordering cost model). On the other hand, decreasing the value m will result in a decrease in the total expected annual cost, the order quantity, the backorder price discount.

Table 5 Solution procedures of Example 2 (L_i *in weeks*)

m	i	L_i	$C(L_i)$	$A(L_i)$	Q_i	p_{x_i}	r_i	$EAC(Q_i, p_{x_i}, L_i)$
-0.2	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	188.49	159.21	77.65	83.71	3,496.33
	2	4	22.4	172.27	151.22	77.52	57.98	3,279.31
	3	3	57.4	160.77	150.92	77.51	44.86	3,239.25 *
-0.5	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	171.23	155.92	77.59	83.71	3,430.60
	2	4	22.4	130.69	142.73	77.37	57.98	3,109.53
	3	3	57.4	101.92	138.73	77.31	44.86	2,995.45 *
-0.8	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	153.79	152.56	77.54	83.71	3,363.46
	2	4	22.4	89.10	133.70	77.22	57.98	2,929.00
	3	3	57.4	43.07	125.36	77.08	44.86	2,728.06 *
-1.0	0	8	0	200.00	166.80	77.78	109.03	3,696.40
	1	6	5.6	142.46	150.28	77.50	83.71	3,317.86
	2	4	22.4	61.37	127.33	77.12	57.98	2,801.54
	3	3	57.4	3.83	115.59	76.92	44.86	2,532.67 *

Table 6 Summary of the optimal solution of Example 2 (L^* *in weeks*)

m	L^*	A^*	Q^*	p_x^*	r^*	$EAC(Q^*,p_x^*,L^*)$	*Saving*(%)
0.0	4	200.00	156.62	77.61	57.98	3.387.38	--
-0.2	3	160.77	150.92	77.51	44.86	3,239.25	4.37
-0.5	3	101.92	138.73	77.31	44.86	2,995.45	11.57
-0.8	3	43.06	125.36	77.08	44.86	2,728.06	19.46
-1.0	3	3.83	115.59	76.92	44.86	2,532.67	25.53

Note: Saving is based on the fixed ordering cost model (i.e., $m = 0$).

5. Conclusions

The primary purpose of this study investigates the inventory system with variable lead time and backorder price discount, and the reduction of lead time accompanies a decrease of ordering cost. We seek to minimize the total expected annual cost by simultaneously optimizing the order quantity Q, backorder discount b, (or backorder price discount p_x), and lead time L. Under the assumption that the lead time demand is normally distributed, an algorithm procedure of finding the optimal solutions is established. Two numerical examples results show that when the reduction of lead time accompanies a decrease of ordering cost and comparing the fixed ordering cost model, indicate that can achieve a significant amount of saving the total expected annual cost, and sensitivity analysis of the optimal solutions with respect to parameters are also indicated.

In future research on this problem, it would be interesting to deal with a mixed stochastic inventory model with the distribution free case where only the mean and standard deviation of lead time demand are known and finite. Moreover, a possible extension of this work may take ordering cost as one decision variable.

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Appendix

For given value of L, we first obtain the Hessian matrix $\mathbf{H}$ is follows:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial Q^2} & \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial Q \partial p_x} \\ \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial p_x \partial Q} & \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial p_x^2} \end{bmatrix}.$$

Then we proceed by evaluating the principal minor determinant of **H**.

The first principal minor determinant of **H** is:

$$\left|H_{11}\right| = \frac{\partial^2 EAC(Q,p_x,L)}{\partial Q^2} = \frac{2A(L)D}{Q^3} + \frac{2DG(p_x)}{Q^3} s\sqrt{L}\, y(k) + \frac{2D}{Q^3} C(L) > 0$$

The second principle minor determinant of **H** is:

$$\left|H_{22}\right| = \begin{vmatrix} \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial Q^2} & \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial Q \partial p_x} \\ \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial p_x \partial Q} & \dfrac{\partial^2 EAC(Q,p_x,L)}{\partial p_x{}^2} \end{vmatrix}$$

$$= \left[\frac{2A(L)D}{Q^3} + \frac{2DG(p_x)}{Q^3} s\sqrt{L}\, y(k) + \frac{2D}{Q^3} C(L)\right]\left[\frac{2Db_0}{Qp_0} s\sqrt{L}\, y(k)\right]$$

$$-\left\{\frac{D^2}{Q^4}\left(\frac{2b_0 p_x}{p_0} - b_0\right)^2 \left[s\sqrt{L}\, y(k)\right]^2\right\}$$

$$= \frac{4D^2 b_0}{Q^4 p_0} s\sqrt{L}\, y(k)\left[A(L) + C(L)\right] + \frac{4D^2 b_0}{Q^4}\left[s\sqrt{L}\, y(k)\right]^2\left(1 - \frac{b_0}{4}\right)$$

$$> 0 .$$

Therefore, it is clear to see that, for given $L \in [L_i, L_{i-1}]$, $EAC(Q,p_x,L)$ is a convex function in (Q,p_x).

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