An Alternative Approach of The Distributional Properties and Moments for A Capability Indices C_{pw}

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ABSTRACT

A unified of capability indices, containing the indices C_p, C_{pk}, C_{pm}, and C_{pmk}, has earlier been defined by Spiring [6] for the case of two-sided specification intervals. In this paper, we investigate several properties of the estimators of the indices for different values of the weight function when the tolerances are symmetric. Again, we study the statistical properties of the natural estimator of C_{pw} (In Spiring [6] a family of capability indices, depending on a weight function, *w*, with symmetric tolerances) and an explicit form of the probability density function of $\hat{C}_{pw}$ under the assumption of normality and in statistical controlled.

Keywords: Capability Index; Sampling Distribution; Moment; Mean Square Error

1. Introduction

Process capability indices (PCIs) are now embraced by a wide variety of industries interested in assessing the ability of a process to meet customer's requirements. When used correctly these indices provide a measure of process performance that in turn can be used in the ongoing assessment of process improvement. The most common process capability indices assume *T* to be the midpoint of the specification limits and include

$$C_p = (USL - LSL)/(6\sigma), \quad (1.1)$$

$$C_{pk} = min\{USL - \mu,\ \mu - LSL\}/(3\sigma) = (1 - k)C_p, \quad (1.2)$$

and $C_{pm} = (USL - LSL)/(6(\sigma^2 + (\mu - T)^2)^{1/2})$, (1.3)

If the characteristic of the process is symmetrically distributed then a shift towards the specification limit, when σ is fixed, which is closest to the target value will give rise to a larger expected proportion of non-conforming than the corresponding shift towards the middle of the specification interval. Therefore, a shift towards the specification limit which is closest to the target value should be considered more serious and give rise to a lower index value than the corresponding shift towards the middle of the specification interval.

Chan et al. [4] considered a generalization of C_{pm} for process with asymmetric tolerances shift one of the two specification limits.

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To obtain a capability index, which is more sensitive than C_{pk} and C_{pm} with regard to departures of the process mean, μ, from the target value, T, Pearn et al. [5] defined a so-called third generation of capability index, C_{pmk}, which is defined by

$$C_{pmk}=\frac{\min(USL-\mu,\mu-LSL)}{3\sqrt{\sigma^2+(\mu-T)^2}}=\frac{d-|\mu-m|}{3\sqrt{\sigma^2+(\mu-T)^2}}=(1-k)C_{pm}. \tag{1.4}$$

Thomas Mathew et al.[9] point out that results of calculations of PCIs should always be qualified via confidence intervals, with a discussion of the impact of the sample size and sampling scheme on the index estimation. It appears difficult to obtain exact confidence intervals for PCIs based on conventional methods, and only approximate or asymptotic confidence intervals are available in the literature for most PCIs(more detail discussion in Kotz and Lovelace [6]). An extensive bibliography of papers on PCIs during the period 1990–2002 is available in Spiring *et al.* [8].

In Spiring [7] a family of capability indices, depending on a weight function, w, with symmetric tolerances is defined as follows:

$$C_{pw}=\frac{USL-LSL}{6\sqrt{\sigma^2+w(\mu-T)^2}}. \tag{1.5}$$

We gain the four indices Cp, Cpk, Cpm, and Cpmk by setting w as follows:

(1) Setting w = 0, then $C_{pw} = C_p$. (1.6)

(2) Setting w = 1, then $C_{pw} = C_{pm}$. (1.7)

(3) Setting $w = \left(\frac{1}{(1-k)^2}-1\right)\frac{1}{p^2}$, where $p=|\mu-T|/\sigma$ is denote a measure of " off- target", then as follows:

(a) If $k=|\mu-m|/d$, then $C_{pw} = C_{pk}$. (1.8)

(b) If $k=|\mu-T|/d=k'$, then $C_{pw} = C'_{pk}$. (1.9)

(4) Setting $w = \left(\frac{1}{(1-k)^2}-1\right)\frac{1}{p^2}+\frac{1}{(1-k)^2}$, then $C_{pw} = C_{pmk}$. (1.10)

In this article, we study the behavior of the indices in this family with respect to different aspects. Moreover, we consider estimators of the proposed class and derive an explicit form of the distribution of the estimator $\hat{C}_{pw}$ for different values of the weight function, such as the behavior of its distribution, and the r-th moment, under the assumption of normality.

Whereas, the demonstrate form of the distribution for the class estimated indices under study could be viewed as an interesting and useful result in the statistical distribution theory, involving a rational function of central and non-central chi-squared distributions in this literature, when deciding on a capability index to be utilized.

2. The sampling distribution of $\hat{C}_{pw}$

Let Xi , i = 1, 2,･･･, n be a random sample from a normal distribution with mean μ and variance σ2 measuring the characteristic under study. Analogously the estimator of Cpm devised by Boyles [2] as

$$\hat{C}_{pw} = \frac{d}{3\sqrt{S_n^2 + w(\bar{X} - T)^2}}, \tag{2.1}$$

where the mean μ is estimated by the sample mean and the variance σ2 is estimated by its maximum likelihood estimator, i.e.,

$$\bar{X} = \sum_{i=1}^{n} X_i / n, \text{ and } S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / n. \tag{2.2}$$

Under the assumption of normality, to derive the distribution of $\hat{C}_{pw}$, where $\hat{C}_{pw}$ is given in (2.1), then this notation the estimator $\hat{C}_{pw}$ becomes:

$$\hat{C}_{pw} = \frac{D}{3\sqrt{K + wY}}, \tag{2.3}$$

where D= $\sqrt{n}$ d/σ, K = n S_n^2 /σ2, Y = n($\bar{X}$ – T)2/σ2, K is independent with Y, and K is a central chi-square distribution with n – 1 degrees of freedom, χ_{n-1}^2, Y is a non-central chi-square distribution with 1 degree of freedom and non-central parameter λ = δ2 = n(μ–T)2/σ2, $\chi_1'^2$ (λ).

By Johnson et al. [5], a non-central chi-square distribution with 1 degree of freedom and non-central parameter λ can be written as a mixture of central chi-square distribution with 1 + 2 j degrees of freedom and corresponding Poisson weights Pj, where

$$P_j(\lambda/2) = \frac{e^{-\lambda/2}(\lambda/2)^j}{j!}, \quad j = 0, 1, 2, \cdots. \tag{2.4}$$

Let f_Y denotes the probability density function of Y can be expressed as

$$f_Y = \sum_{j=0}^{\infty} P_j \, f_{Y_j}(y), \tag{2.5}$$

where Y_j is a central chi-square distribution with 1 + 2 j degrees of freedom.

Using the notation above we state the following theorem.

Theorem 1:

When the characteristic of the process is normally distributed the cumulative distribution function of $\hat{C}_{pw}$, for $w > 0$, is as follows:

$$F_{\hat{C}_{pw}}(x)=\begin{cases}0, & x\le 0,\\ 1-\int_0^{D^2/(9wx^2)} F_K\left(\dfrac{D^2}{9x^2}-wy\right) f_Y(y)\,dy, & x>0.\end{cases} \tag{2.6}$$

Proof:

Using representation (2.3) and conditioning on Y, for $w > 0$, we obtain

$$\begin{aligned} F_{\hat{C}_{pw}}(x) &= P\left(\sqrt{K+wY}\ge \frac{D}{3x}\right)\\ &= 1-\int_0^{\infty} P\left(\sqrt{K+wY}<\frac{D}{3x}\middle| Y=y\right) f_Y(y)\,dy\\ &= 1-\int_0^{\infty} P\left(\sqrt{K+wy}<\frac{D}{3x}\right) f_Y(y)\,dy\\ &= 1-\int_0^{\frac{D^2}{9wx^2}} P\left(K<\frac{D^2}{9x^2}-wy\right) f_Y(y)\,dy. \end{aligned} \tag{2.7}$$

The last equality in (2.7) valid since $P(K<(D^2/(9x^2))-wy)$, for $y > D^2/(9wx^2)$.

Rearranging the last equality in (2.7) we obtain

$$F_{\hat{C}_{pw}}(x) = 1-\int_0^{\frac{D^2}{9wx^2}} F_K\left(\frac{D^2}{9x^2}-wy\right) f_Y(y)\,dy,\ x>0. \tag{2.8}$$

The proof of (2.6) for $x > 0$ and $w > 0$ is thus complete.

To gain the probability density function of $\hat{C}_{pw}$ we first observe that the cumulative distribution function of $\hat{C}_{pw}$ is a continuous function of x.

Theorem 2:

When w > 0, and x > 0 then the characteristic of the process is normally distributed the probability density function of $\hat{C}_{pw}$ becomes:

$$f_{\hat{C}_{pw}}(x)=\frac{2^{1-\frac{n}{2}} e^{\frac{-D^2}{18x^2}}}{x^{n+1}\sqrt{w}}\left(\frac{D}{3}\right)^n \sum_{j=0}^{\infty} P_j(\lambda)\frac{1}{\Gamma\left(\dfrac{n+2j}{2}\right)}\left(\frac{D^2}{18wx^2}\right)^j {}_1F_1(a,b;z), \tag{2.9}$$

where ${}_1F_1(a, b; z) = \Gamma(b)/(\Gamma(a)\Gamma(b-a))\times\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{zt}dt$ is the Kummer confluent hypergeometry function (see Abramorwitz and Stegun, [1]) with parameter $a =(1+2j)/2$, $b = (n+2j)/2$, $z = (w-1)D^2/(18wx^2)$, $D=\sqrt{n}\,d/\sigma$.

Proof :

Taking the derivative of $F_{\hat{C}_{pw}}(x)$ in (2.6) with respect to *x*, we get

$$f_{\hat{C}_{pw}}(x) = \frac{2D^2}{9x^3}\int_0^{\frac{D^2}{9wx^2}} f_K\left(\frac{D^2}{9x^2} - wy\right) f_Y(y)\,dy, \quad x > 0. \tag{2.10}$$

Changing the variable

$$t = (9w\,x^2)\,y/D^2, \tag{2.11}$$

in the integral in (2.10) we can write

$$f_{\hat{C}_{pw}}(x) = \frac{2D^4}{81x^5}\int_0^1 f_K\left(\frac{D^2(1-t)}{9x^2}\right) f_Y\left(\frac{D^2 t}{9wx^2}\right)dt\,,\ x > 0. \tag{2.12}$$

Since *Y* can be expressed as a mixture of central chi-square distribution with 1 + 2 *j* degrees of freedom and corresponding Poisson weights $P_j(\lambda/2)$, and *K* is distributed as χ^2_{n-1}. We can rewrite (2.12) using the expression for the probability density function of the chi-square distribution to obtain the result in (2.9). The proof is thus complete.

Corollary 1:

When the process is on target, *i.e.*, $\mu = T$, which is equivalent to $\lambda = 0$, the cumulative distribution function of $\hat{C}_{pw}$ is obtain from theorem 1 by replacing the probability density function $f_Y(y)$ in (2.6) by

$$f_{Y_0} = \frac{1}{\sqrt{2\pi}\,y}\times e^{\frac{-y}{2}}, \quad y > 0. \tag{2.13}$$

Corollary 2:

When the process is on target, *i.e.*, $\mu = T$, which is equivalent to $\lambda = 0$, the probability density function of $\hat{C}_{pw}$, for $w > 0$, simplifies to

$$f_{\hat{C}_{pw}}(x) = \frac{2^{1-\frac{n}{2}}e^{\frac{-D^2}{18x^2}}}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{D}{3}\right)^n \frac{1}{x^{n+1}\sqrt{\pi w}}\,{}_1F_1\left(\frac{1}{2}\,,\ \frac{n}{2};\ \frac{(w-1)D^2}{18wx^2}\right), \quad x > 0. \tag{2.13}$$

Further, when we have a two-sided specification interval the case when $T = m$ is quite common in practical situations.

Corollary 3:

When the process is on target, *i.e.*, $\mu = T$, or equivalently $\lambda = 0$, with symmetric tolerances, the probability density function of $\hat{C}_{pw}$, for $w > 0$, simplifies as follows:

$$f_{\hat{C}_{pw}}(x) = \frac{2^{1-\frac{n}{2}} e^{\frac{-D^2}{18x^2}}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{D}{3}\right)^n \frac{1}{x^{n+1}\sqrt{\pi w}} \, {}_1F_1\left(\frac{1}{2}, \frac{n}{2}; \frac{(w-1)D^2}{18wx^2}\right), \qquad x > 0. \qquad (2.14)$$

3. Expected Value, Variance, and Mean Square Error(MSE)

By the same technique of $\hat{C}_{pmk}$, the r-th moment of $\hat{C}_{pw}$ can be obtains as

$$E((\hat{C}_{pw})^r) = \left(\frac{D}{3}\right)^r E(K + wY)^{-r/2}$$

$$= \left(\frac{D}{3}\right)^r \sum_{j=0}^{\infty} \frac{e^{\frac{-\lambda}{2}}}{j!} \left(\frac{\lambda}{2}\right)^j E(K + wY_j)^{-r/2}. \qquad (3.1)$$

Under the assumption of normality, Y_j is distributed as χ^2_{1+2j}, K is distributed as χ^2_{n-1}, and K is independent with Y_j. Let $T_j = Y_j/(K+Y_j)$ is distributed as Beta$\left(\frac{1+2j}{2}, \frac{n-1}{2}\right)$, and $W_j = K + Y_j$ is distributed as χ^2_{n+2j} then we have that W_j is independent with T_j. Hence

$$E(K + wY_j)^{-r/2} = E\left(W_j^{\frac{-r}{2}}\right) E(1 + (w-1)T_j)^{-r/2}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^r \times \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} \times {}_2F_1(a,b,c;\, z), \qquad (3.2)$$

where ${}_2F_1(a, b, c; z)$ is the Gaussian hypergeometry function with parameter $a = r/2$, $b = (1+2j)/2$, $c = (n+2j)/2$, and $z = 1 - w$. Then, equation (5.1) becomes:

$$E((\hat{C}_{pw})^r) = \left(\frac{D}{3\sqrt{2}}\right)^r \sum_{j=0}^{\infty} \frac{e^{\frac{-\lambda}{2}}}{j!} \left(\frac{\lambda}{2}\right)^j \times \frac{\Gamma\left(\frac{n+2j-r}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} \times {}_2F_1(a,b,c;\, z). \qquad (3.3)$$

Setting r = 1, and r = 2, we can obtain the expected value and variance of $\hat{C}_{pw}$ as follows:

$$E(\hat{C}_{pw}) = \left(\frac{D}{3\sqrt{2}}\right) \sum_{j=0}^{\infty} \frac{e^{\frac{-\lambda}{2}}}{j!}\left(\frac{\lambda}{2}\right)^{j} \times \frac{\Gamma\left(\frac{n+2j-1}{2}\right)}{\Gamma\left(\frac{n+2j}{2}\right)} \times {}_2F_1\left(\frac{1}{2}, \frac{1+2j}{2}, \frac{n+2j}{2}; 1-w\right), \quad (3.4)$$

$$E(\,(\hat{C}_{pw})^2\,) = \frac{D^2}{9} \sum_{j=0}^{\infty} \frac{e^{\frac{-\lambda}{2}}}{j!}\left(\frac{\lambda}{2}\right)^{j} \times \frac{1}{n+2j-2} \times {}_2F_1\left(1, \frac{1+2j}{2}, \frac{n+2j}{2}; 1-w\right). \quad (3.5)$$

The variance of $\hat{C}_{pw}$ is now obtained as

$$Var(\hat{C}_{pw}) = E(\,(\hat{C}_{pw})^2\,) - (\,E(\hat{C}_{pw})\,)^2. \quad (3.6)$$

Since the estimator $\hat{C}_{pw}$ is biased the mean square error of the estimator might be more relevant to investigate than the variance.

$$MSE\,(\hat{C}_{pw}) = Var(\hat{C}_{pw}) + (\,E(\hat{C}_{pw}) - C_{pw}\,)^2$$

$$= E(\,(\hat{C}_{pw})^2\,) + (C_{pw}\,)^2 - 2\,C_{pw}\,E(\hat{C}_{pw}), \quad (3.7)$$

4. Discussion

As a general rule, we have a two-sided specification interval the case when $T = M$ is quite common in practical situations. In the class of indices studied we are looking for an index that is sensitivity to departure from the target value, T, especially in the case when b is bigger.

According to Vännman [10] suggestion for obtaining meaningful indices is suited to the following two criterions.

1. Only indices will small bias and small MSE will be considered, when the process is on target.
2. Among the possible w-value obtained, indices will be chosen with regard to their sensitivity to departures from the target value in the sense that the expected value of the estimator of the index ought to be sensitivity to departures from the target value, especially in the case of large b. And then, the MSE also will be taking into consideration, when the process is not on target.

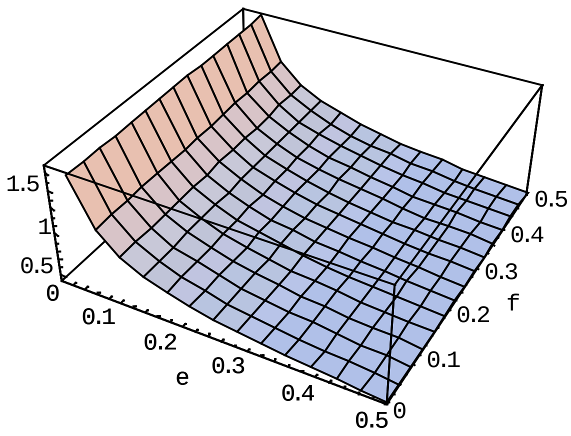
To explore the behavior of the estimator for different values of w the expected values, the relative biased values, and the mean square errors were calculated, using Maple V software, for different values of the parameters

$$w, \quad n, \quad a = \frac{|\mu - T|}{\sigma}, \text{ and } \quad b = \frac{d}{\sigma},$$

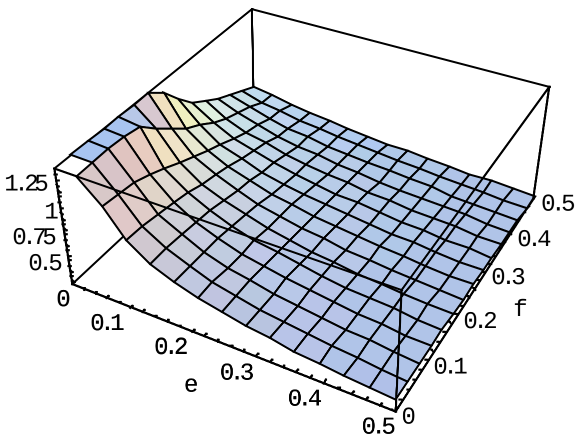
we did calculate using $a = 0(0.5)2$, $b = 2(1)6$, $n = 10(20)50$ and $w = 0(1)5$.

If it is of interest to have a capability index that is very sensitive with regard to departures of the process mean, μ, from the target value, T, then the values of w in (1.5) should be large. In Figure 1 some plots $C_{pw}(w)$ are given for some w values, using Mathematica software.

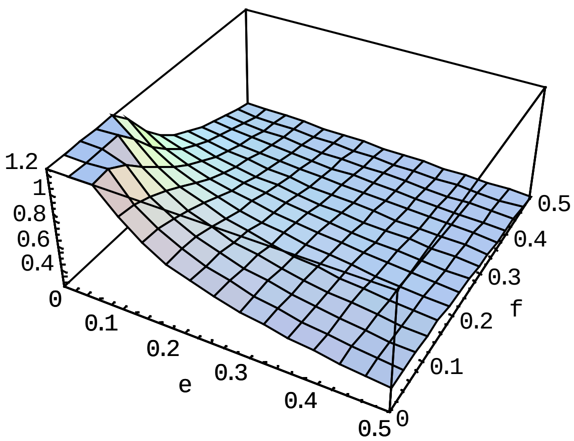
The indices have been expressed in the two variables $e = 1/b$ and $f = |\mu - T|/d$ in Figure 1, and the surface describing the index to make easy comparisons of the indices. From Figure 1 we can see how the sensitivity, with regard to departure the process mean, μ, from the target value, T, rely on w.



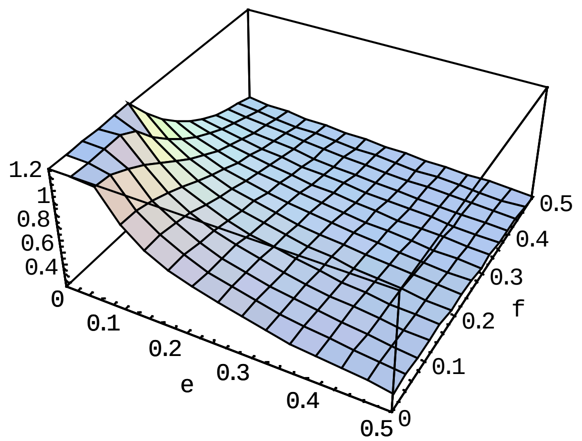
Cpw(0)



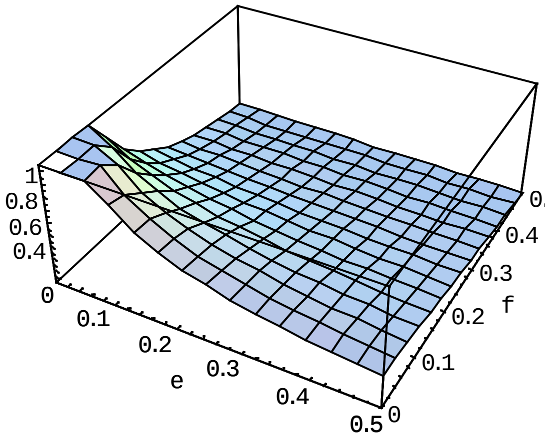
Cpw(1)



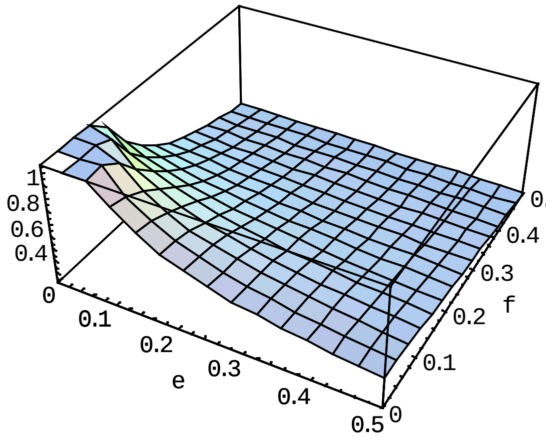
Cpw(2)



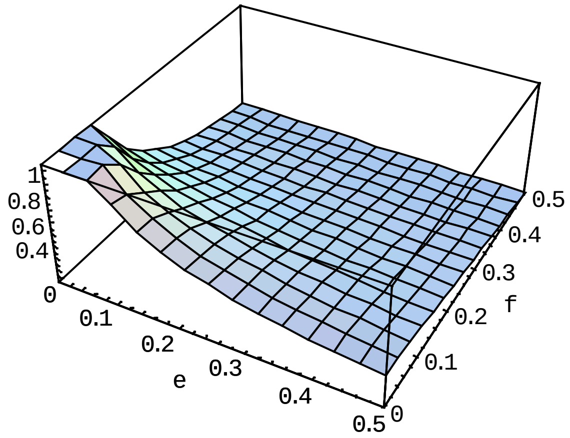
Cpw(3)



Cpw(4)



Cpw(5)



Cpw(6)

Figure 1: The capability indices *Cpw*(*w*), *w*=0(1)6 as surface plots with $b = d/\sigma = 1/e$, $f = |\mu - T|/d$.

In Table 1 we can see, when the process is not on target, $a > 0$, the bias of $C_{pw}(4)$ is positive, for n = 10, 20, 30. And then, in Table 1 the relative bias of $C_{pw}(w)$, when $a = 0$, can be positive or negative, is given for n = 10(20)50, b = 2(1)6, and w = 0(1)6.

Table 1 : The Relative bias of $C_{pw}(4)$, when n = 10, 20, 30.

		a				
	b	0	0.5	1	1.5	2
	2	-0.018281	0.037205	0.023496	0.009368	0.004285
	3	-0.027421	0.055807	0.035246	0.014051	0.006428
n = 10	4	-0.036562	0.074410	0.046994	0.018735	0.008571
	5	-0.045702	0.093011	0.058744	0.023419	0.010713
	6	-0.054843	0.111618	0.070488	0.028103	0.012856

Table 1 : The Relative bias of $C_{pw}(4)$, when n = 10, 20, 30. (cont.)

n = 20	2	-0.014709	0.020110	0.011046	0.004418	0.002064
	3	-0.022064	0.030165	0.016571	0.006627	0.003096
	4	-0.029418	0.040258	0.022094	0.008836	0.004127
	5	-0.036773	0.050275	0.027618	0.011045	0.005159
	6	-0.044127	0.060330	0.033078	0.013253	0.006191
n = 30	2	-0.011521	0.015288	0.008131	0.002892	0.001360
	3	-0.017281	0.022932	0.012196	0.004338	0.002039
	4	-0.023042	0.030575	0.013939	0.005785	0.002719
	5	-0.028802	0.038219	0.018291	0.007231	0.003399
	6	-0.034563	0.045863	0.021949	0.008677	0.004079

We see from Table 2 that, when the process is on target, the relative bias is negative and quite large in absolute value when w is large. The relative bias of C_{pw} in absolute value is decreasing, when n or b is increasing and fixed w.

Table 2: The Relative bias of $C_{pw}(w)$, when the process is on target.

		w						
	b	0	1	2	3	4	5	6
n = 10	2	0.102288	0.055815	0.024987	0.001216	-0.018281	-0.034858	-0.049297
	3	0.153432	0.083722	0.037480	0.001824	-0.027421	-0.052288	-0.073945
	4	0.204576	0.111630	0.049973	0.002432	-0.036562	-0.069717	-0.098593
	5	0.255720	0.139537	0.062467	0.003040	-0.045702	-0.087146	-0.123242
	6	0.306864	0.167445	0.074960	0.003648	-0.054843	-0.104575	-0.147890
n = 30	2	0.029586	0.017266	0.006576	-0.002921	-0.011521	-0.019394	-0.026667
	3	0.044380	0.025899	0.009864	-0.004381	-0.017281	-0.029091	-0.040001
	4	0.059173	0.034533	0.013152	-0.005842	-0.023042	-0.038789	-0.053335
	5	0.073966	0.043166	0.016441	-0.007302	-0.028802	-0.048486	-0.066668
	6	0.088759	0.051799	0.019729	-0.008762	-0.034563	-0.058183	-0.080002
n = 50	2	0.017300	0.010213	0.000958	-0.002296	-0.007924	-0.013206	-0.018194
	3	0.025950	0.015319	0.001437	-0.003444	-0.011885	-0.019810	-0.027291
	4	0.034600	0.020426	0.001915	-0.004592	-0.015847	-0.026413	-0.036389
	5	0.043250	0.025532	0.002394	-0.005740	-0.019809	-0.033016	-0.045486
	6	0.051900	0.030638	0.002873	-0.006887	-0.023771	-0.039619	-0.054583

In Table 3 the values, when $a = 0$, of the MSE($C_{pw}(w)$), are given for $n = 10(20)50$, $b = 2(1)6$, and $w = 0(1)6$. We see from Table 2 that, when $n = 10(20)50$, the smallest values of the mean square error are obtained for $w = 1, 2, 3, 4$, but the MSE does not vary too much close to these values. We also see from Table 3 that large values of w cause large MSE and hence indices with large values of w are not suitable to use.

For example, choosing the indices with the four smallest mean square errors for $b = 2(1)6$ and $n = 10(20)50$ from Table 3 will give us $w = 1, 2, 3, 4$.

Table 3 : The MSE of $C_{pw}(w)$, when the process is on target.

		w						
	b	0	1	2	3	4	5	6
$n = 10$	2	0.054092	0.036691	0.032632	0.032077	0.032997	0.034641	0.036662
	3	0.121707	0.082555	0.073422	0.072173	0.074243	0.077941	0.082490
	4	0.216369	0.146765	0.130527	0.128307	0.131988	0.138562	0.146648
	5	0.338076	0.229321	0.203949	0.200479	0.206231	0.216504	0.229138
	6	0.486830	0.330222	0.293687	0.288690	0.296972	0.311765	0.329958
$n = 30$	2	0.009934	0.008724	0.008405	0.008566	0.009043	0.009724	0.010545
	3	0.022352	0.019630	0.018910	0.019273	0.020346	0.021879	0.023726
	4	0.039736	0.034897	0.033618	0.034263	0.036171	0.038896	0.042180
	5	0.062088	0.054527	0.052528	0.053536	0.056517	0.060775	0.065907
	6	0.089407	0.078519	0.075641	0.077091	0.081384	0.087516	0.094906
$n = 50$	2	0.005302	0.004901	0.004480	0.004895	0.005126	0.005473	0.005903
	3	0.011930	0.011028	0.010080	0.011013	0.011532	0.012315	0.013281
	4	0.021209	0.019606	0.017920	0.019579	0.020502	0.021894	0.023610
	5	0.033139	0.030634	0.028000	0.030592	0.032034	0.034209	0.036891
	6	0.047720	0.044113	0.040320	0.044053	0.046130	0.049260	0.053123

5. Sensitivity Analysis

Next, we compare the indices, among those obtained above, with regard to their sensitivity to departures from the target value. From Table 4 we see that $w = 1, 2$ will give us the index is least sensitivity to departures from the target value when b is large, and hence we exclude that case.

Table 4 : The expected value of $C_{pw}(w)$, when n = 10.

	w = 1					*w* = 2				
	a					*a*				
b	0	0.5	1	1.5	2	0	0.5	1	1.5	2
2	0.722482	0.644322	0.501133	0.385603	0.306734	0.691653	0.582389	0.414817	0.305528	0.206387
3	1.083722	0.966483	0.751700	0.578405	0.460101	1.037480	0.873583	0.622226	0.474766	0.342717
4	1.444963	1.288644	1.002266	0.771206	0.613468	1.383307	1.164778	0.823052	0.633022	0.456956
5	1.806204	1.610805	1.252833	0.964008	0.766835	1.729133	1.454873	1.025133	0.763820	0.571194
6	2.167445	1.932966	1.503399	1.156809	0.920202	2.074960	1.747166	1.234578	0.872412	0.685433
	w = 3					*w* = 4				
2	0.667883	0.540112	0.357398	0.249822	0.189802	0.648386	0.508609	0.321638	0.220186	0.165976
3	1.001824	0.810148	0.536104	0.374734	0.284703	0.972579	0.762914	0.482459	0.330279	0.248964
4	1.335765	1.080209	0.714795	0.499645	0.379603	1.296772	1.017219	0.643279	0.440372	0.331952
5	1.669707	1.350261	0.893494	0.624556	0.474504	1.620964	1.271522	0.804100	0.550465	0.414940
6	2.003648	1.620313	1.072198	0.749467	0.569405	1.945157	1.525832	0.964915	0.660558	0.497927

Therefore, we finish with the two indices corresponding to w = 3 and 4, which are nearly equivalent with regard to mean square errors and sensitivity to departures from the target value. But again, we see from Table 1, the estimator $C_{pw}(4)$ is rather more sensitivity to departures from the target value but will induce a small and negative bias, for n = 10(20)50, while $C_{pw}(3)$ is almost unbiased. And then, we see from Table 5 that the mean square error of the $C_{pw}(3)$ and $C_{pw}(4)$ are almost equivalent.

Table 5 : The MSE of $C_{pw}(w)$, when n = 10.

	w = 1					*w* = 2				
	a					*a*				
b	0	0.5	1	1.5	2	0	0.5	1	1.5	2
2	0.036691	0.028202	0.013004	0.004948	0.002016	0.032632	0.027672	0.007853	0.001779	0.010859
3	0.082555	0.063454	0.029258	0.011133	0.004536	0.073422	0.062261	0.017672	0.018053	0.002343
4	0.146765	0.112807	0.052014	0.019792	0.008063	0.130527	0.110686	0.041548	0.032099	0.004456
5	0.229321	0.176261	0.081272	0.030925	0.012599	0.203949	0.175939	0.072012	0.011120	0.006962
6	0.330222	0.253816	0.117032	0.044532	0.018143	0.293687	0.249045	0.093494	0.059377	0.010025
	w = 3					*w* = 4				
2	0.032077	0.029273	0.010738	0.002821	0.000926	0.032997	0.030599	0.010071	0.002354	0.000734
3	0.072173	0.065895	0.024154	0.006347	0.002083	0.074243	0.068847	0.022659	0.005296	0.001652
4	0.128307	0.117122	0.042954	0.011284	0.003703	0.131988	0.122394	0.040282	0.009414	0.002937
5	0.200479	0.183004	0.067115	0.017631	0.005786	0.206231	0.191245	0.062939	0.014710	0.004588
6	0.288690	0.263525	0.096635	0.025387	0.008332	0.296972	0.275377	0.090641	0.021182	0.006607

It is important to note that, when n is increasing, the mean square error for all estimators is decreasing. Also we can be seen from Table 6, which gives the square root of the mean square error for $C_{pw}(4)$ for some parameter values. In this paper, we investigated the statistical properties of the natural estimator of C_{pw}, the exact distribution, r-th moment and the behavior of the expected value and the mean square error, when deciding which of the capability indices to use, assuming that the process is normally distributed.

Table 6 : The square root of the MSE for $C_{pw}(4)$ and corresponding index, for a = 0, 1, 2, b = 3, 5 and n = 10, 30, 50.

		$(\text{MSE}(C_{pw}(4)))^{1/2}$			
a	b	$C_{pw}(4)$	n = 10	n = 30	n = 50
0	3	1.000	0.272	0.143	0.107
1	3	0.447	0.151	0.073	0.054
2	3	0.243	0.041	0.022	0.017
0	5	1.667	0.454	0.238	0.179
1	5	0.745	0.251	0.121	0.090
2	5	0.404	0.068	0.036	0.028

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