Finite Deformation of 2-D Thin Circular Curved Laminated Beams

Chiu-Wen Lin

Abstract

An analytical method is derived for obtaining the finite deformation of 2-D thin curved laminated beams. The general solutions are expressed by fundamental geometric quantities. As the radius of curvature is given, the fundamental geometric quantities can be calculated to obtain the closed form solutions of the axial force, shear force, bending moment, rotation angle, and deformed or un-deformed displacement fields. The closed-form solutions of the circular curved laminated beams under pure bending moment case are presented. It shows the consistency of the results of present study with those by ANSYS.

Keywords：Finite deformation, Curved laminated beams, Variable curvatures, Analytical solutions, Non-linear behavior.

Chiu-Wen Lin, Lecturer, Department of General Education Center, HIT.
投稿日期：990427 接受刊登日期：990628

二維圓形薄疊層曲樑之有限變形研究

林秋文

摘 要

本研究應用解析方法，分析研究二維圓形薄疊層曲樑之有限變形；其一般通解以曲樑之基本幾何特性值表示之。當曲率半徑確定時，則曲樑之基本幾何特性值可以計算出；並藉以求得曲樑之剪力、軸向力、彎矩、旋轉角、變形位移場與未變形位移場等物理量之閉合型式解。本文發表了懸臂圓形薄曲樑承受純彎矩作用之閉合型式解，並與有限元素法套裝分析軟體 ANSYS 分析結果比較；結果非常一致。

關鍵詞：有限變形理論、疊層曲樑、解析解、非線性行為。

林秋文：修平技術學院通識教育中心講師

1. Introduction

The rod theory is one of the most developed parts of the elasticity theory. The finite deformation of rods in space, are always related nonlinear geometric behavior. There are two approaches which are very common. One is three dimensional rods theory. The rod is treated as a three dimensional elastic body to which the methods of three dimensional elasticity theory are applied （Green and Naghdi [1]）. The other is one dimensional director theory. The rod is treated as a curve（Green and Naghdi [2]）. Naghdi [3], Green [4] showed the nonlinear behavior of rods in both ways. Green [5] showed some relationship between two approaches.

Due to the complexity of mathematical models, most studies have to adopt some kind of simplification, such as small displacement, small shearing strain, small rotation or small shearing effect. By using finite element method, Li. [6,7] derived a finite deformation theory based on total Lagrangian description for 2-D and 3-D beams of zero Poisson's ratio without all the simplifications. Some studied the finite deformation under dynamic loading. Oguibe [8] investigated the numerical study of the elastic plastic response of multilayer aluminum cantilever beams subjected to an impulse loading. The numerical results were compared with the experimental results. Attard [9] studied finite strain of an isotropic hyper-elastic Hookean beam. He used an appropriate strain energy density. The shear effect was included. The solution was also applied to stability behavior design of a helical spring. Mauget [10] applied Lagrangian coordinates to derive an isotropic constitutive law for a large displacement formulation of woods. Toi [11] used total Lagrangian approach for the super-elastic large deformation analysis of a shape memory alloy helical springs.

Most studies focus on straight rods. Only few investigate curve rods. Atanackovic [12] analyzed the finite deformation of a circular ring under uniform pressure. Brush [13] derived a finite deformation stability equation for circular ring under various pressures. He also investigated the stability of nonlinear equilibrium equations for fluid pressure loading. Due to the complexity of mathematical models, analytical solutions are very limited. Timoshenko [14] showed the large deformation of an Elastica. It also showed the stability of a straight beam of large deformation. In this paper, we apply the tangent slope coordinate theory by Lin [15, 16, 17, 18] and lamination theory by Herakovich [19] to study finite deformation 2-D of curved laminated beams with circular curvature.

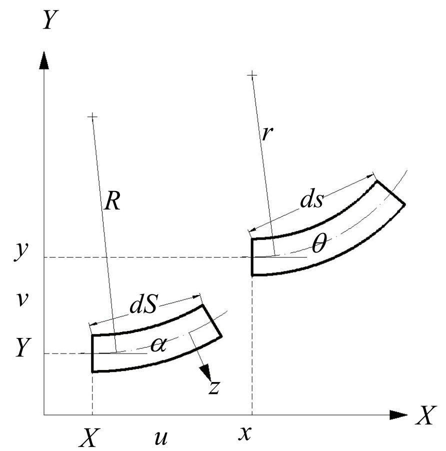

Fig. 1. Deformation of length element

2. Fundamental Equations

2.1 Displacement Field and Kinematics Relationship

Consider an elastic curved laminated beam with variable curvature whose axis lies on a 2-D plane. Assume the rod is made of elastic material such that the stress is linear to the strain even for finite deformation of the beam. Since the strain is finite, the displacement of a point, the extension of the axis and the rotation angle of any cross section are not necessarily small. To simplify the analysis, assume cross sections do not change the shape and size and the cross section is always orthogonal to the axis in the deformed state.

To describe the laminated curve beam on a 2-D reference configuration, the un-deformed length element *dS* after deformation become the deformed length element *ds*. The coordinate of end point （*X,Y*） in the un-deformed state deforms to （*x,y*） shown as Fig. 1. At the un-deformed state, the tangent slope angle at （*X,Y*） is denoted by α. At the deformed state, the tangent slope at （*x,y*） is denoted by θ. The deformation at （*X,Y*） is denoted by （ *u,v* ） where *u* is the horizontal displacement, and *v* is the vertical displacement. Hence

$$x = X + u, \quad y = Y + v. \tag{1}$$

The rotation angle φ can be found by

$$\varphi = \theta - \alpha.. \tag{2}$$

Since the strain at the centroid axis is defined by ε= （*ds-dS*）/*dS*, or

$$ds = (1+\varepsilon)dS. \tag{3}$$

As in the case of in-extensional curved beam, $\varepsilon=0$. For any length element *dS*, there is a corresponding radius of curvature *R*, such that

$$dS = Rd\alpha. \tag{4}$$

Here the radius of curvature *R* does not have to be a constant. Most well known curves can be determined by specifying the radius of curvature, such as circle, ellipse, parabola, cycloid, hyperbola, centenary, spiral curves, etc.

For the deformed length element *ds*, the corresponding radius of curvature is denoted by *r*, i.e.

$$ds = rd\theta. \tag{5}$$

At a distance z from centroid axis, the un-deformed length element is denoted by

$$dS_z = (R + z)d\alpha \cdot \tag{6}$$

And the deformed length element is

$$ds_z = (r + z)d\theta \cdot \tag{7}$$

The strain at a distance of z defined by

$$\varepsilon_z = \frac{ds_z - dS_z}{dS_z} = \frac{\varepsilon + z\dfrac{d\varphi}{dS}}{1 + \dfrac{z}{R}}. \tag{8}$$

Equation （8） can be simplified to

$$\varepsilon_z = \varepsilon + z\frac{d\varphi}{dS}. \tag{9}$$

Here assume $z<<R$ so that $z/R<<1$ can be neglected. In other words, the curved beam is slender in the sense that dimension of cross section is much less than the dimension of radius of curvature.

2.2 Constitutive Relationship

Consider a thin curved laminated beam composed of layers. The layers are arranged on the centroid plane even the fiber can be in different direction. Here the effect of shear deformation is neglected. The stress-strain relationship for a layer along axis direction is

$$\sigma = Q_{11}\varepsilon_z \cdot \tag{10}$$

Where σ is the normal stress component along tangent direction and Q_{11} is the elastic stiffness coefficient of the laminated material. The coefficient Q_{11} is expressed by

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \cdot \tag{11}$$

Where E_1 is the longitudinal Young's modulus along fiber direction, ν_{12} is the longitudinal Poisson's ratio, and ν_{22} is the transverse Poisson's ratio. For a n-layers of laminated beam, the stress-strain relationship of K_{th} layer due to rotation transformation

$$[\sigma]_k = [\overline{Q}_{11}]_k[\varepsilon]_k \cdot \tag{12}$$

Where

$$\overline{Q}_{11} = Q_{11}\cos^4\gamma + 2(Q_{12} + 2Q_{66})\cos^2\gamma\sin^2\gamma + Q_{22}\sin^4\gamma \tag{13}$$

With

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \cdot \tag{14}$$

In the expression, E_2 is the transverse Young's modulus perpendicular to fiber direction, G_{12} is longitudinal shear modulus, and γ is the angle between tangential direction and fiber direction, respectively. The coefficients $Q_{11}, Q_{12}, Q_{66}, Q_{22}$ are also the elastic stiffness coefficients.

Assume that the width of the cross section is b and the total thickness is $h=nt$ （Fig. 2.）. At each cross section there are n-layers from bottom to the top. At the bottom it is the first layer and at the top it is n-th layer.

At any cross section, the resultant force and moment are obtained by integrating the stress in each layer through the thickness. Here the bending moment is positive if it intends to decrease the radius of curvature of the centroid axis. The notation and sign convention of axial force N, moment M together with shear force Q, external distributed tangential force q_α and radial force q_R are shown in Fig. 3.

The axial force N and moment of the laminate are obtained by integration of stresses in each layer through the thickness,

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \varphi' \end{bmatrix}. \qquad (15)$$

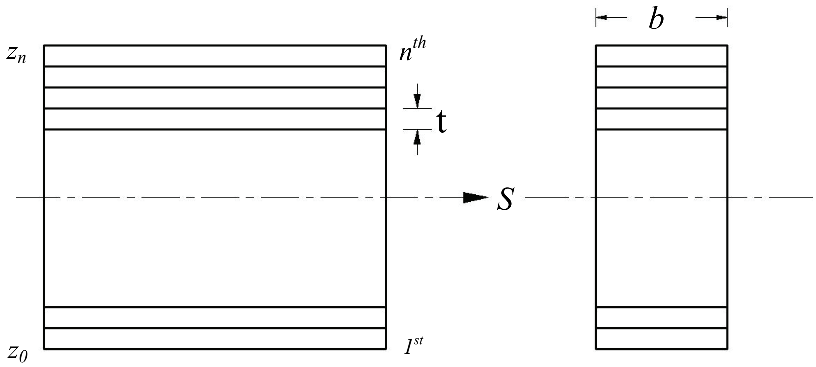

Fig. 2. Layers of laminated curved beam

Where

$$A_{11} = \sum_{k=1}^{n} b \int_{z_{k-1}}^{z_k} \left[\overline{Q}_{11}\right]_k dz,$$

$$D_{11} = \sum_{k=1}^{n} b \int_{z_{k-1}}^{z_k} \left[\overline{Q}_{11}\right]_k z^2 dz, \qquad (16)$$

$$B_{11} = \sum_{k=1}^{n} b \int_{z_{k-1}}^{z_k} \left[\overline{Q}_{11}\right]_k z dz.$$

The varible φ' denotes $d\varphi / dS$, the A_{11} represents the in-plane stiffness, the B_{11} defines the bending-stretching coupling and the D_{11} is the bending stiffness. Since in engineering application, most used materials are carbon fiber and glass fiber. The thickness of carbon fiber is almost the same. Therefore in application it is convenient to used same thickness. However even if different thicknesses are used as the curved laminated beam, the equations of Eq.（16） are still valid. In the case of symmetric laminated, the integral $B_{11}=0$. Then the strain and curvature change will be decoupled, i.e.

$$N = A_{11}\varepsilon \quad , \quad M = D_{11}\varphi'. \qquad (17)$$

Here it is noted that if the longitudinal Young's modulus and the transverse Young's modulus are equal in each layer, that implies Q_{11} = $Q_{22} = E'/(1-\nu^2)$, then in the laminate $A_{11}=E'A/(1-\nu^2)$, $B_{11} = 0$, D_{11}

= $E'/(1-\nu^2)$ =*EI*. Hence the anisotropic material will reduce to be an isotropic material.

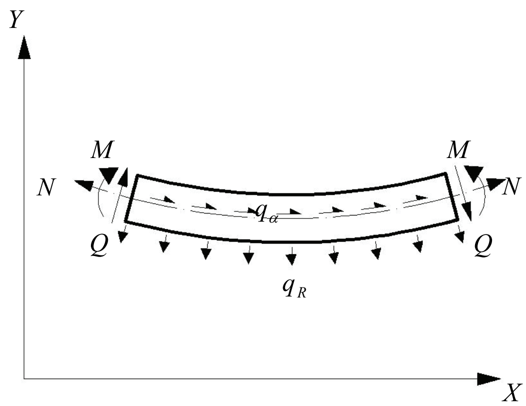

Fig. 3. Sign convention of forces and moment

2.3 Kinetics

To demonstrate the force balance, there are several ways, such as in *X*-*Y* plane and expressing all quantities in terms of *X*-*Y* components. The other choice is to express all quantities in terms of reference configuration, expressing all quantities in axial and shear components or tangent and normal components. Another choice is in terms of spatial configuration or deformed configuration.

The force balance in the reference configuration can be expressed by

$$\frac{dN}{dS}+\frac{Q}{R}=-q_\alpha, \quad -\frac{N}{R}+\frac{dQ}{dS}=-q_R, \quad \frac{dM}{dS}=Q. \tag{18}$$

The three equations show the balance of forces along tangential direction, radial direction and moment. The equilibrium equations can be obtained by taking free body of a curved element. These equations are the same as the force balance equation in small deformation [15], since the effect of finite deformation does not effect equilibrium equation in the reference configuration.

The equilibrium equations can also be expressed by deformed configuration,

$$\frac{dN}{ds}+\frac{Q}{r}=-\frac{q_\alpha}{1+\varepsilon},$$
$$-\frac{N}{r}+\frac{dQ}{ds}=-\frac{q_R}{1+\varepsilon}, \quad \frac{dM}{ds}=\frac{Q}{1+\varepsilon};$$
$$dx=(1+\varepsilon)\cos\theta dS, \quad dy=(1+\varepsilon)\sin\theta dS; \tag{19}$$
$$N=A_{11}\varepsilon+B_{11}(1+\varepsilon)\frac{d\varphi}{ds},$$
$$M=B_{11}\varepsilon+D_{11}(1+\varepsilon)\frac{d\varphi}{ds}.$$

There are three configurations to describe the balance equations. Later it will show that to solve the system equations, some configuration are needed to combined and solved. The complicated term arises from *1+* ε term. This term is retained due to the conservative field of external loads. However this term also induces a nonlinear effect. Note that, from Eq. （19c）, there is no stability term. Axial induced moment is not included. Hence the system equation should promote no buckling mode. It shows the finite deformation.

Here assume the external distributed loads q_α, q_R change direction but not in magnitude. There are several ways to describe equilibrium equations. So far there are five equations ： three forces Eqs. （19a, b, c） and two constitutive equations Eqs. （19f, g） for N, Q, M, and ε, φ. To complete the analysis of finite deformation, the displacements and Constitutive Relationships are included. Once taking the integrals of Eqs. （19d, e）, from Eq. （1）, the displacements u, v can be found.

3. Applications

3.1 Pure Bending Solutions

Here to demonstrate the analytical solution of a finite deformation of angle-ply curved laminated beam, consider a curved beam symmetric with respect to Y axis, and extending from $\alpha=-\beta$ to $\alpha=\beta$. The un-deformed radius of curvature may be different. Hence this curved beam can be many kinds of curved beams. A couple of concentrated moments are applied at both ends. Assume that the origin（un-deformed state） is located at $\alpha=0$ （Fig. 3a）.

Here the deformed configuration is used. In the absence of distributed loads q_α, q_R in Eqs. （19a, b）

$$\frac{dN}{ds}+\frac{Q}{r}=0，\ -\frac{N}{r}+\frac{dQ}{ds}=0， \tag{20}$$

$$\frac{dM}{ds}=Q；$$

$$\frac{dx}{ds}=\cos\theta，\ \frac{dy}{ds}=\sin\theta；$$

$$N=A_{11}\varepsilon+B_{11}(1+\varepsilon)\frac{d\varphi}{ds}，$$

$$M=B_{11}\varepsilon+D_{11}(1+\varepsilon)\frac{d\varphi}{ds}$$

By using changing of variable Eqs.（20a, b） can be induced to

$$\frac{dN}{d\theta}+Q=0，\ -N+\frac{dQ}{d\theta}=0\cdot \tag{21}$$

Taking derivative of Eq.（21a）, combining with Eq.（21b） and eliminating the variable Q, the equation yields,

$$\frac{d^2N}{d\theta^2}+N=0\cdot \tag{22}$$

The solution of N is

$$N=A_1\cos\theta+A_2\sin\theta\cdot \tag{23}$$

Where A_1, A_2 are two constants to be determine by suitable boundary conditions. Taking derivative of Eq.（23）, with the help of Eq.（21a）, the variable Q is

$$Q=A_1\sin\theta-A_2\cos\theta\cdot \tag{24}$$

At the un-deformed ends $\alpha=\beta$, $N(\pm\beta)=Q(\pm\beta)=0$. At the deformed ends, $\theta=\theta_\beta=\beta+\varphi(\beta)$, $N(\beta+\varphi(\beta))=Q(\beta+\varphi(\beta))=0$, or

$$\begin{aligned}&N=A_1\cos(\beta+\varphi(\beta))+A_2\sin(\beta+\varphi(\beta))=0\\&,\\&Q=A_1\cos(\beta+\varphi(\beta))-A_2\sin(\beta+\varphi(\beta))=0\\&.\end{aligned} \tag{25}$$

Where $\varphi(\beta)$ is the deformed rotation angle at $\alpha=\beta$. It is still an unknown and needed to find. The solution of Eqs.(25) is

$A_1 = A_2 = 0$.

Hence

$$N = Q = 0. \tag{26}$$

With the help of Eq. (26), integrate Eqs. (20c), once to obtain

$$M = A_3. \tag{27}$$

Where A_3 is a constant.

At the free end, the boundary condition is

$$M(\beta + \varphi(\beta)) = M_0. \tag{28}$$

Hence the solution of M is

$$M = -M_0. \tag{29}$$

Therefore Eqs. (20f,g) yield

$$A_{11}\varepsilon + B_{11}(1+\varepsilon)\frac{d\varphi}{ds} = 0, \quad B_{11}\varepsilon + D_{11}(1+\varepsilon)\frac{d\varphi}{ds} = M_0. \tag{30}$$

The solution of ε and $(1+\varepsilon)\, d\varphi/ds$ is

$$\varepsilon = \frac{-M_0 B_{11}}{A_{11}D_{11} - B_{11}^2}, \quad (1+\varepsilon)\frac{d\varphi}{ds} = \frac{M_0 A_{11}}{A_{11}D_{11} - B_{11}^2}. \tag{31}$$

The solution Eq. (31) show that even under pure bending moment, the strain at the centroid line is still deformed. This is caused by material properties. For a symmetric curved laminated beam, $B_{11} = 0$ and $\varepsilon = 0$.

Then there is no deformation along centroid line. It becomes inextensible. Since the coefficients of A_{11}, B_{11} and D_{11} are all positive, and denominator $A_{11}D_{11}-B_{11}^2$ is normally positive, ε is then negative. In other words, the curved laminated beam becomes longer under the positive pure bending case. Eq. (31b) is expression in terms of un-deformed state for the sake of integration. Integrating Eq. (31b) once to obtain

$$\varphi = \int_0^{\alpha} \frac{A_{11}M_0}{A_{11}D_{11} - B_{11}^2} R d\alpha + \varphi_0. \tag{32}$$

Where φ_0 is a constants, determined by boundary conditions. Here the deformation angle is expressed in term of reference configuration. Due to symmetric, $\alpha=0$, $\varphi(0)=0$. Hence $\varphi_0=0$.

The Eq. (32) implies

$$\varphi = \frac{A_{11}M_0}{A_{11}D_{11} - B_{11}^2} S. \tag{33}$$

The deformed angle can be expressed in the reference configuration. Furthermore the deformed shape can be evaluated by the deformed coordinates x, y. The integral of Eqs. (20d, e),

$$x = \int_0^{\alpha}\left(1 - \frac{M_0 B_{11}}{A_{11}D_{11} - B_{11}^2}\right)\cos\left(\alpha + \frac{A_{11}M_0}{A_{11}D_{11} - B_{11}^2} S\right) R(\alpha)\, d\alpha, \quad y = \int_0^{\alpha}\left(1 - \frac{M_0 B_{11}}{A_{11}D_{11} - B_{11}^2}\right)\sin\left(\alpha + \frac{A_{11}M_0}{A_{11}D_{11} - B_{11}^2} S\right) R(\alpha)\, d\alpha. \tag{34}$$

The integration constants are vanished due to the symmetric conditions of $\alpha=0$, $x=0$, $y=0$. There is no deformation at the origin.

The integrals Eq. (4) depend on the form in the integral. Since the arc length is calculated based on the given curvature.

Once the radius of curvature is assigned, the arc can be evaluated. Then the integrals of Eq.（34） can be found. However, only some given radius of curvature can be carried out to explicit form （instead of integral form）, due to the difficulty of integral. In this analysis, the results show at least some curves can be calculate explicitly, some are still not, even all curves can be evaluated by numerical techniques. For an angle-ply symmetric curved laminated beam, $B_{11}=0$, $\varepsilon = 0$.

The deformed coordinated Eq.（34） can be simplified to

$$x = \int_0^\alpha R(\alpha)\cos\left(\alpha + \frac{M_0}{D_{11}}S\right)d\alpha,$$
$$y = \int_0^\alpha R(\alpha)\sin\left(\alpha + \frac{M_0}{D_{11}}S\right)d\alpha \qquad (35)$$

Here the procedures show that no need to calculate the deformations u, v but directly evaluate the deformed coordinates x, y.

Consider a cantilever angle-ply symmetric curved laminated beam with variable curvature. The curve starts from fixed end $\alpha=0$ to free end $\alpha=\beta$ shown as Fig. 3a. The origin is located at $\alpha=0$. A concentrated moment M_0 is applied at the end of $\alpha=\beta$. It is equivalent to a free curved beam which is symmetric with respect to y-axis. The curved beam starts from free end $\alpha=-\beta$, to $\alpha=\beta$. A pairs of concentrated moment M_0 are applied at both ends shown as Fig. 3b. Due to symmetry, only the portion from $\alpha=0$ to $\alpha=\beta$ is needed to be considered. Due to fixed end or symmetry, at $\alpha=0$ the boundary conditions are

$$\varphi = 0,\ u = 0,\ v = 0,$$
$$\theta = 0,\ x = 0,\ y = 0. \qquad (36)$$

At the free end （$\alpha=\beta$）, the axial force, shear force and moment are

$$N(\beta) = 0,\ Q(\beta) = 0,\ M(\beta) = M_0. \qquad (37)$$

Substituting Eqs.（37） into Eqs.（23, 24）, the constants can be found $A_1=A_2=0$, $A_3=M_0$. Substituting Eq.（37a） into Eqs.（34a, b）, the deformed angle of slope becomes

$$\theta = \alpha + \lambda S,\quad \lambda = \frac{M_0}{D_{11}}. \qquad (38)$$

The Eq.（35） can be expression by

$$x = \int_0^\alpha R\cos(\alpha + \lambda S)d\alpha,$$
$$y = \int_0^\alpha R\sin(\alpha + \lambda S)d\alpha. \qquad (39)$$

In Eqs.（39）, the constants from integrating vanish because at $\alpha=0$, $x=y=0$. Once the radius of curvature of the curve beam is specified, the deformed coordinates can be found. The displacements u, v with the help of Eq.（1） is then

$$u = \int_0^\alpha 2R\sin\left(\alpha + \frac{1}{2}\lambda S\right)\sin\left(\frac{1}{2}\lambda S\right)d\alpha,$$
$$v = \int_0^\alpha 2R\sin\left(\frac{1}{2}\lambda S\right)\cos\left(\alpha + \frac{1}{2}\lambda S\right)d\alpha. \qquad (40)$$

Note that the radius of curvature r after deformation can be calculated by taking derivatives of Eqs.（39）. It can be simplified to

$$r = \frac{R}{1+\lambda R}. \qquad (41)$$

It is seen that if the curve is a circle, or R is a constant, then r is a constant as well and only changes magnitude. The magnitude factor is $1/(1+\lambda R)$.

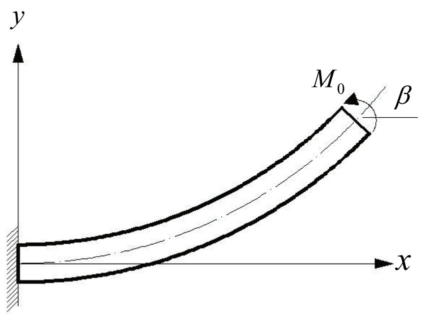

Fig. 3a. A cantilever curved beam under a concentrated moment M_0.

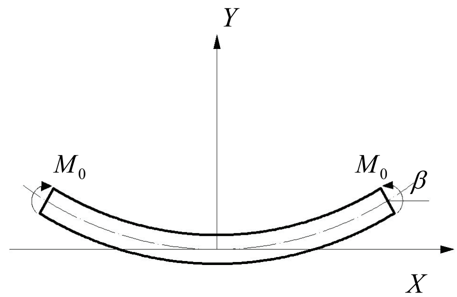

Fig. 3b. Simple curved beams under a pair of moment M_0.

However for a curve of variable curvature, R is not a constant. The deformed radius of curvature changes to another function by $1/(1+\lambda R)$. Therefore the curve changes type as it is deformed.

3.2 *Finite deformation of circular laminate curve beam under pure bending moments*

For an angle-ply circular laminated beam, choose the radius as the characteristic radius, i.e.

$$R = 1. \qquad (42)$$

It is equivalent to $X=\sin\alpha$, $Y=1-\cos\alpha$ in parametric form. The solution of displacements u, v in Eq. (40) can be integrated to yield

$$\begin{aligned} u &= \frac{\sin(1+\lambda)\alpha}{1+\lambda} - \sin\alpha, \\ v &= -\frac{\lambda}{1+\lambda}\cos\alpha - \frac{\cos\alpha}{1+\lambda}. \end{aligned} \qquad (43)$$

The deformed coordinates from Eqs. (39) are

$$\begin{aligned} x &= \frac{1}{1+\lambda}\sin(1+\lambda)\alpha, \\ y &= \frac{1}{1+\lambda}\left[1-\cos(1+\lambda)\alpha\right]. \end{aligned} \qquad (44)$$

The coordinates of Eqs. (44) are a circle in parametric form. The radius of curvature is $1/(1+\lambda)$. In other words as the moment M_0 increases, the radius of curvature decreases by the scale of $1/(1+\lambda)$. Since the strain at the centroid axis is zero, the circumference can be calculated to close the circular curve. The required moment to close the circle is

$$M_{o_{close}} = \frac{D_{11}}{R_0}\left(\frac{\pi}{\beta} - 1\right).$$

Or, the required λ_{close} to close the circle is

$$\lambda_{close} = \frac{\pi}{\beta} - 1. \qquad (45)$$

For instance to close a quadrant symmetric curve laminated beam（from $-\pi/4$ to $\pi/4$）, the required moment is $\lambda_{close}=3$. The deformed curves of a quarter circular at various closed moments are shown in Fig. 4.

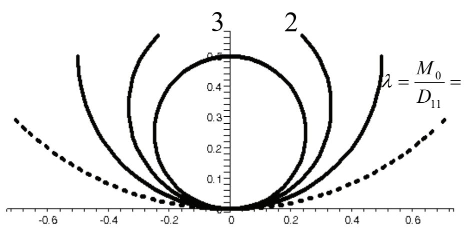

Fig. 4. Symmetric angle-ply circular curved laminated beams from $-\pi/4$ to $\pi/4$ under a couple of moment M_0.

The final angle versus close moment of angle-ply laminates are shown in Fig. 5.

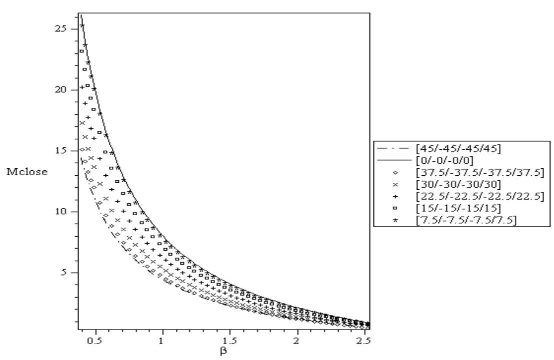

Fig. 5. The final angle versus close moment of various 4-layers stacking sequences.

Consider a cantilever angle-ply symmetric circular curved laminated beam subjected to a pure bending moment $M_0=D_{11}/R_o$, the curve starts from fixed end $\alpha=0$ to free end $\alpha=\pi/2$. Beams cross-section consist with *2* layers of carbon fiber T300/5208 [$-\theta, \theta$] Material properties are $E_1=132$Gpa, $E_2=10.8$Gpa and $\nu_{12}=0.24$, $G12=5.65$Gpa, $M=1.867572156N \cdot m$ each layer thickness $h=1mm$, the width is $b=20mm$.

The fiber orientation angles versus equivalent λ_{eq} of different stacking numbers are shown in Fig. 6. And the fiber orientation angles versus bending stiffness of different stacking numbers are shown in Fig. 7.

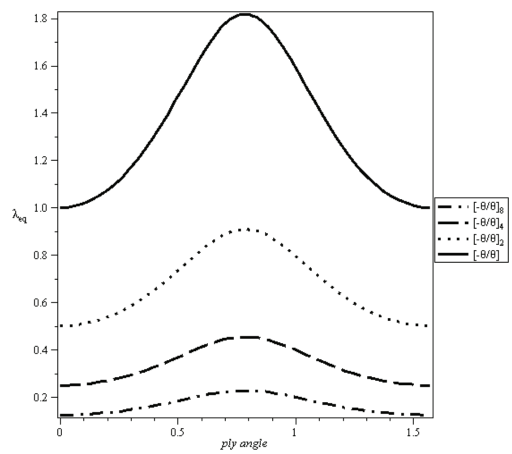

Fig. 6. The fiber orientation angles versus equivalent λ_{eq} of different stacking numbers（$\gamma=0$ to $\gamma=\pi/2$）

Figure 6, shows the cantilever angle-ply symmetric circular curved laminated beam under same apply moment. The different fiber orientation angles stacking sequences from *0* to $\pi/2$, it can be seen the equivalent λ of curved laminated beam will increases to the tip at $\pi/4$. Fig. 7, shows the cantilever angle-ply symmetric circular curved laminated beam.

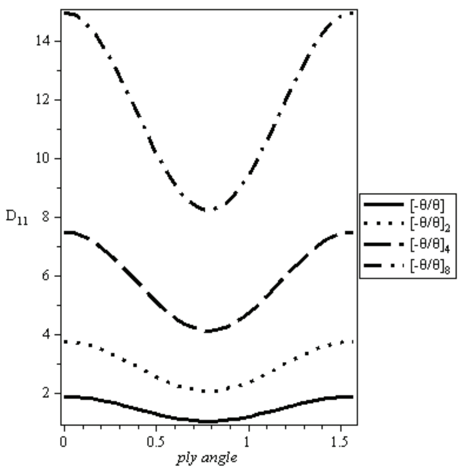

Fig. 7. The fiber orientation angle γ versus bending stiffness of different layer numbers （$\gamma=0$ to$\gamma=\pi/2$）

The different fiber orientation angles and layer numbers from *0* to $\pi/2$, it can be seen the bending stiffness of curved laminated beam, D_{11}, exhibit minima at $\gamma=\pi/4$. With the helps of Eq.（13）, the bending stiffness D_{11} can be expressed in terms of the fiber orientation angles and layer numbers *n* using Eq.（46）,

$$D_{11}=\left(\frac{b\cdot h^3}{12}\overline{Q}_{11}\right). \quad (46)$$

Noted that, laminate consisting of an equal number of equal-thickness layers at +θ and –θ fiber orientations are called angle-ply laminates, such laminates are specially orthotropic. Because such laminate do not exhibit coupling between in-plane extensional and shear response. For a symmetric curved laminated beam, $B_{11}=0$. The deformation determine by D_{11}.

The deformations of different stacking sequences are shown in Fig. 8.

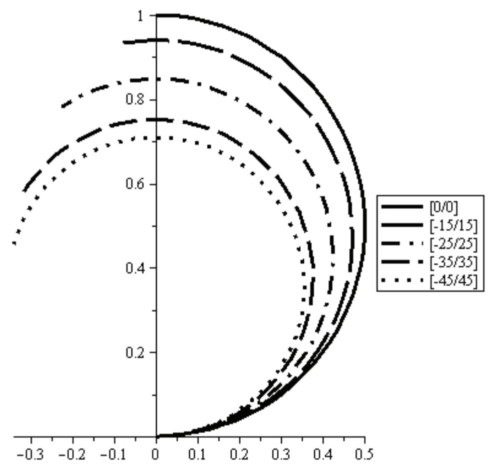

Fig. 8. The deformation shapes of different stacking sequences

Figure 9, shows the free end deformed *x*-coordinates and *y*-coordinates with various λ from *0* to *10*. As λ increases to *7, 15,….* The cantilever angle-ply symmetric circular curved laminated beam will always spin to fixed end （x,y）=（$0,0$）.

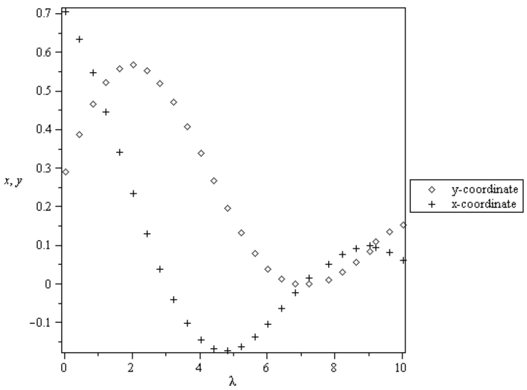

Fig. 9. The free end deformed coordinate of cantilever circular curved beam （from 0 to $\pi/4$）

The deformation shapes shown as Fig. 10.

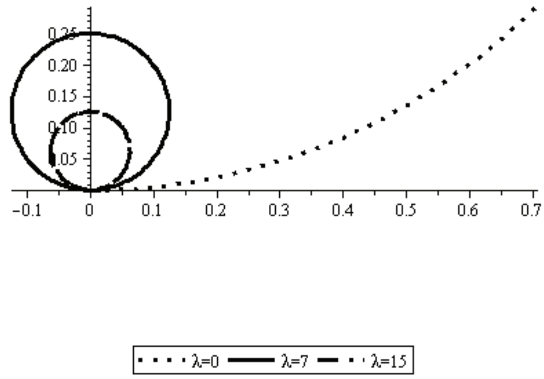

Fig. 10. The deformation of circular curved beam （from 0 to $\pi/4$） under various λ.

3.3 ANSYS results

Consider a cantilever angle-ply symmetric circular curved laminated beam subjected to a pure bending moment $M_0=D_{11}/R_o$, the curve starts from fixed end $\alpha=0$ to free end $\alpha=\pi/2$. Beams cross-section consist with $2n$ layers of carbon fiber T300/5208 [0_{2n}], Material are $E_1=132$Gpa, $E_2=10.8$Gpa and $\nu_{12}=0.24$, $G12=5.65$Gpa, each layer thickness $t=1mm$, the total thickness is $h=2nt$ and the width is $b=20mm$. Using ANSYS large deformation static analysis, the various R/h versus deformed displacements u is shown in Table 1.

Table 1. The various R/h versus deformed displacements u.

R/h	2500	1250	1000	500	200	100	10
u	0.999737	0.999987	0.999987	0.999751	0.999751	0.999751	0.999751
Error（%）	0.0263%	0.0013%	0.0013%	0.0249%	0.0249%	0.0249%	0.0249%

Where $h/b=1/10$.

It shows the consistency of the results of present study with those by ANSYS.

4. Conclusions

This paper has presented an analytical method for obtaining the finite deformation of 2-D circular curved laminated beam. The rod axis is either inextensible or extensible. The curved beam is slender in the sense that dimension of cross section is much less than the dimension of radius of curvature. To derive the analytical method for the general solutions, one can introduce the coordinate system defined by the radius of centroidal axis and the angle of tangent slope.

The general solutions expressed by fundamental geometric quantities form a set of equations having seven unknown constants. The seven constants can be directly determined by suitable boundary conditions. As the radius in terms of the tangent slope angle is given, the fundamental geometric quantities can be calculated to obtain the closed form solutions of the axial force, shear force, bending moment, rotation angle, and displacement fields at any cross-section of circular curved laminated beams. These results of the applications indicate that the closed-form general solutions derived by the analytical method would be valid for in-plane thin angle-ply circular curved laminated beams. Thus the analytical method would be useful to engineers attempting to obtain the exact expressions for thin angle-ply circular curved laminated beams in engineering applications. Especially, helical spring and clockwork spring production and process.

References

[1] Green A. E., Naghdi P.M., Wenner M. L., *One the theory of rods I. Derivations from the three-dimensional equations*, Proc. Roy. Soc. Lond. A. 337, 1974, 451-483.

[2] Green A. E., Naghdi P.M., Wenner M. L., *One the theory of rods II. Developments by direct approach*, Proc. Roy. Soc. Lond. A. 337, 1974, 485-507.

[3] Naghdi P.M., *Finite deformation of elastic rods and shells*, Proc. Iutam Symp. On Finite Elasticity, Leihigh Univ., Aug. 1980, 47-103.

[4] Green A. E., Naghdi P. M., *Non-isothermal theory of rods, plates and shells*. Int. J. Solids Strucs. 1970, 6, 209-244.

[5] Green A. E., Laws N., *Remarks on the theory of rods*, J. Elasticity, 1973, 3,

179-184.

[6] Li M., *The finite deformation theory for beam, plate and shell Part I. The two-dimensional beam theory*, Computer Methods Appl. Mech. Engrg. 146 （1997） 53-63.

[7] Li M., *The finite deformation theory for beam, plate and shell Part III. The three dimensional beam theory and FE formulation.*

[8] Oguibe C.N., Webb D. C., Large deflection analysis of multilayer cantilever beams subjected to impulse loading, Computers and Structures, 78, 2000, 537-547.

[9] Attard M. M., *Finite strain – beam theory*, International J. Solids and Structures, 40, 2003, 4563-4584.

[10] Mauget B., Perré P., *A large displacement formulation for anisotropic constitutive laws*, Eur. J. Mech. A/ Solids 18, 1999, 859-877.

[11] Toi Y., Lee J., Taya M., *Finite element analysis of super-elastic large deformation behavior of shape memory alloy helical springs*, Computers and Structures 82, 2004, 1685-1693.

[12] Atanackovic T. M., *Stability theory of elastic rods*, World Scientific, 1997.

[13] Brush D. O., Almroth B. O., *Buckling of bars, plates and shells*, McGraw-Hill New York, 1975, 121

[14] Timoshenko S. P., *Theory of elastic stability*, McGraw-Hill, 1961,76.

[15] Lin K. C., Huang S. H., *Static Closed-form Solutions for In-plane Thick Curved Beams with Variable Curvatures*, J. Solid Mechanics and Material Engineering, 2007, 1, 1026-34.

[16] Lin K. C., Huang S. H., *Static Closed-form Solutions for In-plane Shear Deformable Curved Beams with Variable Curvatures*, J. Solid Mechanics and Material Engineering, 2007, 1, 1362-73.

[17] Lin, K.C. and Hsieh, C.M., T*he closed form general solutions of 2-D curved laminated beams of variable curvatures*, *Composite Structures* Vol.79 （2007）, pp.606-618.

[18] Lin C. W., *Finite Deformation of 2-D Thin circular curved beams*, Hsiuping Journal, Vol. 19 （2009）, pp.203-216.

[19] Herakovich C. T., *Mechanics of Fibrous Composites*, John Wiley & Sons, 1998.