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Charts with Asymmetric Sample Size to Monitor the Process Means

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***Abstract*—This paper proposed $\bar{x}$ charts with asymmetric sample size to monitor the asymmetric properties of the process, with an objective function of Tauchi's quality loss during out-of-control period. Numerical results indicate when the magnitudes of upward and downward shift are not large and the difference between them is very large, $\bar{x}$ charts with asymmetric sample size is more appropriate for dealing with asymmetric quality loss function than that with symmetric sample size. Moreover, the larger loss coefficient rate is, the larger upward shift frequency is, the more efficient the proposed charts are.**

Keywords- controlcha rt; asymmetric loss; magnitude of shift; asymmetric sample size

I. INTRODUCTION

There are many common causes and assignable causes which make industrial production process off-target, which results in some loss. In practice, it is often assumed that the loss function corresponding to controlling quality characteristics is symmetric about the target in the production process. However, some off-target loss functions are not symmetric, such as one part over the ceiling, you can reprocess, but if less than the minimum allowed size it will be set aside, the loss is clearly asymmetric.

A process is often influenced by many assignable causes and the frequency , the shift directions and the shift magnitudes assignable causes occur are not different. For example, one cause makes the process mean upward shift, and the mean values increase; while another makes the process mean downward shift, and the mean values decrease .However, only when one assignable cause affects the process, the process may have two types of shifts.

When the cost function is asymmetric, the target need to be adjusted to make the average cost be the smallest[1,2,3]。Yang discussed an asymmetric loss function in the economic design of the chart, because of the complexity of economic models[4], only from the view of an experimental design, and gave the impact of the loss coefficient about the optimal economic design results.

This paper presents an asymmetric sample size（ASS）$\bar{x}$ chart to monitor the asymmetry properties in the production process, and then calculates the quality loss of the process during its out-of-control state. Matched symmetric sample size $\bar{x}$ chart (SSS) is compared with it in the end. The main idea of asymmetric sample size chart is: if the current sample falls within the upper control zone, then the next sample size is selected as n_1 ; if the current sample falls with the lower control zone, then the next sample size is n_2 .

II. ASYMMETRIC QUALITY LOSS FUNCTION

A. *Taguchi Quality Loss Function*

This is inherent in Taguchi's(1986)definition of quality, which states that "quality is the loss a product causes to society after being shipped ,other than any losses caused by its intrinsic functions." The loss includes not only direct losses, such as air pollution, noise pollution and chemical leaks, etc., but also indirect losses, such as the customers' not being satisfied with the products, as well as the resulting loss from market and sales , also the additional insurance costs.

Taguchi Hyun quantified the deviations from requirements in terms of monetary units by using a quadratic loss function given by

$$l(x) = k(x-m)^2$$

where random variable x is the quality characteristic, k is a cost-related numerical constant, and m , $0 < m < +\infty$,is target value.

Mr Taguchi, Hyun said average quality loss of the product $E[l(x)]$ as quality level.

B. *Asymmetric Quality Loss Function*

It is may be asymmetric loss caused when performance indicators of products are sometimes bigger than the target value and those smaller than the target value .

Consider asymmetric loss function

$$l(x) = \begin{cases} k_1(x-m)^2, x \le m \\ k_2(x-m)^2, x > m \end{cases}$$

where $k_1, k_2 > 0$, m is target value of quality characteristic.

If the distribution density function of the quality characteristic is $f(x;\mu)$, μ is process mean, the average quality loss is:

$$L(\mu)=E[l(x)]=\int_{-\infty}^{+\infty} l(x)f(x;\mu)dx$$
$$=(k_1-k_2)\int_{-\infty}^{m}(x-m)^2 f(x;\mu)dx$$
$$+k_2[D(x)+(E(x)-m)^2]$$

Let $r=k_2/k_1$, r is the rate of loss coefficient. But the size of k_1, k_2 does not affect the analysis to the specific process.

Given $k_1=1, k_2=r$, we have the formula as follows:

$$L(\mu)=k_1\{(1-r)\int_{-\infty}^{m}(x-m)^2 f(x;\mu)dx + r[\sigma^2+(\mu-m)^2]\} \quad (1)$$

III. THE $ASS\ \bar{x}$ CHART DESIGN

A. Assumptions

(1) Suppose that x is a normally distributed random variable with mean μ and variance σ^2. The probability density function of x is

$$f(x;\mu)=\frac{1}{\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)} \quad \sigma>0 \quad (2)$$

Let $E(x)=\mu$, $D(x)=\sigma^2$ and the probability density function be incorporated into equation (1),we obtain

$$L(\mu)=k_1\sigma^2\{(1-r)\frac{1}{\sqrt{2\pi}}\frac{m-\mu}{\sigma}e^{-(m-\mu)^2/(2\sigma^2)}$$
$$+(1-r)[1+(\frac{m-\mu}{\sigma})^2]\Phi(\frac{m-\mu}{\sigma})$$
$$+r[1+(\frac{m-\mu}{\sigma})^2]\} \quad (3)$$

(2) The mean of a process that is in -control state is μ_0, and the standard deviation is σ, fixed through the whole process.

(3) The target mean μ_0 of the process is equal to the target of quality characteristic, m.

(4) Suppose that the probability which the process mean shifts from μ_0 to $\mu_0+\delta_U\sigma$ is p_U, the the probability which the process mean shifts from μ_0 to $\mu_0-\delta_d\sigma$ is $p_L=1-p_U$ .where $\delta_U\geq 0, \delta_d\geq 0$. Thus ,the loss is given by

$$L(\mu_0+\delta_U\sigma)=k_1\sigma^2\{(1-r)\frac{1}{\sqrt{2\pi}}(-\delta_U)e^{-\delta_U^2/2}$$
$$+(1-r)[1+\delta_U{}^2]\Phi(-\delta_U)+r[1+\delta_U^2]\} \quad (4)$$

$$L(\mu_0-\delta_d\sigma)=k_1\sigma^2\{(1-r)\frac{1}{\sqrt{2\pi}}\delta_d e^{-\delta_d^2/2}$$
$$+(1-r)[1+\delta_d{}^2]\Phi(\delta_d)+r[1+\delta_d{}^2]\} \quad (5)$$

Obviously,when $r>1$, $\delta_U=\delta_d=\delta$, we can obtain $L(\mu_0+\delta\sigma)>L(\mu_0-\delta\sigma)$.

B. The $ASS\ \bar{x}$ Chart Design Units

Now we design a chart with asymmetric sample sizes to monitor the changes of the process mean.

The standardized sample means $Z=(\bar{x}-\mu_0)/\sigma_{\bar{x}}$ are plotted on an $\bar{x}$ chart with control limits $\pm\gamma$, where $\gamma>0$, $\sigma_{\bar{x}}=\sigma/\sqrt{n}$ .When the sample point falls outside the control limits, a signal is given, then the search for the assignable cause is undertaken. Either it is proved a falsealarm or we remove the assignable cause. The control zone is divided into two parts as follows:

Upper control zone $I_1=[0,\gamma)$,

Lower control zone $I_2=(-\gamma,0)$.

The rules of choosing sample size are: if the current sample point falls within the upper control zone I_1, the next sample size is denoted by n_1, and if the sample point falls within the lower control zone I_2, then the next sample size is denoted by n_2 ,of course, if the sample point falls outside the control limits, an signal is given, and assignable causes should be searched. It might be preferable that the first sample be large to give additional protection against start-up problems.

Using standardized sample mean, and plot it at the control charts whose control limits have nothing to do with sample size. If the process is in control i.e. $\mu=\mu_0$, the probability which the sample falls in each zone is independent of sample size. However, when the process mean shifts, using different sample sizes means that the control charts will have different efficiency.

For the same shift magnitude, because the loss due to the upward shift is larger, the value of n_1 should be bigger than n_2 ,which can quickly detect the shifts causing larger quality loss.

C. The ASS $\bar{x}$ Chart Performance Measure

When the sampling interval between two samples is fixed, the statistical efficiency of charts can be measured by the average number of sample points that are plotted （ARL） and the average number of the individual products （ANI） before an out-of control condition. In the design of control charts, the speed that shifts are detected in the process conflicts with low false-alarm rate, that is, we wish the larger in–control ARL and the smaller out -of -control ARL, but the two goals are contradictory. If a deviation from the normal operating behavior in the process occurs, in order to avoid excessive defective products, we should quickly detect the shifts. However, too frequently stopping production to check the problem might lead to incorrect adjustments for the process which makes the staff lose confidence in the use of control charts.

The ARL value associated with a shift $\mu-\mu_0=\delta\sigma$ is called the "out-of-control" value denoted by $ARL(\delta)$. It represents the expected number of sample points until an out-of-control signal is obtained. Let $ANI(\delta)$ denotes the corresponding average numbers of individual products until an out-of-control signal.

We calculate the values of $ARL(\delta)$ and $ANI(\delta)$ using Markov chains. Consider an absorbing Markov chain with two transient states{1，2}:

State 1 : A sample mean falls within the upper control zone;

State 2 : A sample mean falls within the lower control zone;

State 3 : A sample mean falls outside the control zone. Obviously, state 3 is an absorbing state.

The transition probability matrix is given by

$$\mathbf{Q}=\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

When $\mu=\mu_0+\delta\sigma$， then

$$p_{i1}=\Phi(\gamma-\delta\sqrt{n_i})-\Phi(-\delta\sqrt{n_i}) \; ;$$

$$p_{i2}=\Phi(-\delta\sqrt{n_i})-\Phi(-\gamma-\delta\sqrt{n_i})$$

$i=1\,,2$.

From the elementary properties of Markov chains, we obtain

$$ARL(\delta)=B^T(I-Q)^{-1}1 \qquad (5)$$

$$ANI(\delta)=B^T(I-Q)^{-1}S \qquad (6)$$

where B is the starting probability vector, $B^T=(a,b)$, $a+b=1$. is the identity matrix of order 2, $1=(1,1)^T$, $S=(n_1,n_2)^T$.

When the process is in-control state, equation （5） reduces to be

$$ARL(0)=1/q_0$$

where $q_0=p(Z>\gamma \; or Z<-\gamma \,|\, Z\sim N(0,1))$.

So we can get

$$ARL_2=p_U\times ARL(\delta_U)+p_L\times ARL(-\delta_d)$$

$$ANI_2=p_U\times ANI(\delta_U)+p_L\times ANI(-\delta_d)$$

The average sample size at average sampling point is as follows:

$$\bar{n}=\frac{ANI_1}{ARL_1} \qquad (7)$$

In practice it is more usual that the shift in the process mean doesn't occur at the beginning but at some random time in the future. For such case, then the total quality loss during an out-of –control period is given by

$$TL=p_U\times g\times h\times[ARL(\delta_U)-0.5]\times L(\mu_0+\delta_U\sigma) + p_L\times g\times h\times[ARL(-\delta_d)-0.5]\times L(\mu_0-\delta_d\sigma)$$

where g is productivity and h is the sampling interval between samples.

IV. COMPARISON WITH SYMMETRIC SAMPLE SIZE $\bar{x}$ CHARTS

In order to express the advantage to use ASS $\bar{x}$ chart, we compare it and the symmetric sample size （SSS） $\bar{x}$ chart matched with it. If the two charts had the same control limits, then they should have the same in-contol average run length. Then during an in-control period, we use symmetric sample size, that is, when $n_1=n_2=n_0$, $ARL(0)=1/q_0$. Design an ASS $\bar{x}$ chart to make an average sample size $E(n)$ in control state equal to the fixed sample size n_0 , then during an in-control period the two controls have the same false- alarm rate.

Note that the probability the sample falls within every zone is not relevant to the used samples when it is in-control state, so

$$E(n\,|\,\mu=\mu_0)=\frac{n_1p_{01}+n_2p_{02}}{1-q_0} \qquad (8)$$

Where

$$p_{01}=p(0<Z<\gamma\,|\,Z\sim N(0,1))$$

$$p_{02}=p(-\gamma<Z<0\,|\,Z\sim N(0,1)),$$

$$p_{01}+p_{02}=1-q_0,$$

thus

$$n_1+n_2=2n_0 \qquad (9)$$

Suppose that the shift in the process mean doesn't occur at the beginning, but at some time in the future. Thus each element of the starting probability vector is the proportion of the time spent on every transient state, that is

$$a = 0.5,\quad b = 0.5$$

Let $CP = \dfrac{TL_A}{TL_S}$,

which denotes that the ratio of quality loss TL_A used in asymmetric sample size charts (ASS) to quality loss TL_S used in symmetric sample charts(SSS). If $CP < 1$ implies that it is of good efficiency to use an asymmetric sample size $\bar{x}$ chart . the smaller the value of CP, the better efficiency; If $CP > 1$ implies that it is of good efficiency to use a symmetric sample size $\bar{x}$ chart. Here is part of the numerical results.

From table 1, 2, 3, 4,we can obtain :

(1) If the magnitude of upward shift is far less than the magnitude of downward shift in the process mean, then we use $n_1 > n_2$,the quality loss could be reduced. For example, when $\delta_u / \delta_d = 0.3$, using $(n_1, n_2) = (9,1)$ can make $CP \in [0.47, 0.54]$ .Obviously, results can be improved.

(2) The larger the ratio of loss coefficient r, the better the efficiency of the ASS $\bar{x}$ chart.

(3) The bigger the value of p_U, the better the efficiency of the ASS $\bar{x}$ chart. For example, when $\delta_u / \delta_d = 0.3$, $r = 2$, $p_U = 0.3, 0.5, 0.7, 0.9$, using $(n_1, n_2) = (9,1)$ can obtain CP =0.63,0.54,0.49,0.47.

(4) The better performence of ASS $\bar{x}$ charts are usually in cost of larger average sample size.

when $\delta_U > \delta_d$, the performance of ASS $\bar{x}$ charts are bad. We will discuss how the various values of r, p_U, n_1 affect CP.

From table 6 and other tables, we can obtain that the magnitude of upward and downward shifts are the key factor whichdetermines the effects of using asymmetric sample sizes. If both of them are large, then the use of asymmetric sample sizes does not make sense.

Ⅳ. CONCLUSIONS

When the production conditions are not meeted with the assumption Shewhart chart based on, we can consider the use of the control chart of asymmetric sample size .Moreover, the more severe the extent of asymmetry, the greater frequency the upper shifts happen, the better the use of ASS $\bar{x}$ charts. If the magnitude of an upward shift is far less than that of an downward shift, it is recommended to use asymmetric sample size charts ; If the magnitude of an upward shift and that of an upward shift are both very large, it is appropriate to use symmetric sample size.

Assume the control limits and the sample size are pre-set, which would make the model more simple and easy to be implemented.

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TABLE I. THE RATIO OF QUALITY LOSS ABOUT ASS $\bar{x}$ CHARTS AND THE MATCHED SSS $\bar{x}$ CHARTS DURING AN OUT-CONTROL PERIOD ($p_U = 0.3, p_L = 0.7$)

CP	$r = 2$	$r = 5$	$r = 10$
	(9,1)(8,2)(6,4)	*(9,1)(8,2)(6,4)*	*(9,1)(8,2)(6,4)*
δ_u / δ_d 0.1	***0.86 0.88 0.96***	***0.86 0.88 0.96***	***0.86 0.88 0.96***
0.3	***0.63 0.60 0.81***	***0.56 0.57 0.81***	***0.53 0.57 0.81***
0.5	***1.18 0.79 0.84***	***0.84 0.68 0.82***	***0.71 0.64 0.81***
1.0	***3.39 1.72 1.05***	***2.67 1.57 1.04***	***2.25 1.49 1.04***
2.0	***3.31 1.58 1.04***	***2.47 1.36 1.02***	***1.95 1.22 1.01***

TABLE II. THE RATIO OF QUALITY LOSS ABOUT ASS $\bar{x}$ CHARTS AND THE MATCHED SSS $\bar{x}$ CHARTS DURING AN OUT-CONTROL PERIOD ($p_U = 0.5, p_L = 0.5$)

CP	$r = 2$	$r = 5$	$r = 10$
	(9,1)(8,2) (6,4)	(9,1)(8,2) (6,4))	(9,1) (8,2) (6,4)
δ_u / δ_d 0.1	0.85 0.88 0.96	0.85 0.88 0.96	0.85 0.88 0.96
0.3	0.54 0.57 0.81	0.50 0.56 0.81	0.49 0.55 0.81
0.5	0.83 0.68 0.81	0.66 0.62 0.80	0.60 0.60 0.80
1.0	2.71 1.58 1.04	2.17 1.48 1.04	1.92 1.43 1.04
2.0	2.53 1.37 1.02	1.86 1.19 1.01	1.55 1.11 1.01

TABLE III. THE RATIO OF QUALITY LOSS ABOUT ASS $\bar{x}$ CHARTS AND THE MATCHED SSS $\bar{x}$ CHARTS DURING AN OUT-CONTROL PERIOD ($p_U = 0.7, p_L = 0.3$)

CP	$r = 2$	$r = 5$	$r = 10$
	(9,1) (8,2) (6,4)	(9,1) (8,2) (6,4)	(9,1) (8,2) (6,4))
δ_u / δ_d 0.1	0.84 0.88 0.96	0.84 0.88 0.96	0.84 0.88 0.96
0.3	0.49 0.55 0.81	0.48 0.55 0.81	0.47 0.55 0.80
0.5	0.66 0.62 0.80	0.58 0.59 0.80	0.56 0.59 0.80
1.0	2.20 1.48 1.04	1.90 1.42 1.04	1.75 1.39 1.04
2.0	1.90 1.20 1.01	1.50 1.09 1.03	1.33 1.05 1.00

TABLE IV. THE RATIO OF QUALITY LOSS ABOUT ASS $\bar{x}$ CHARTS AND THE MATCHED SSS $\bar{x}$ CHARTS DURING AN OUT-CONTROL PERIOD ($p_U = 0.9, p_L = 0.1$)

CP	$r = 2$	$r = 5$	$r = 10$
	(9,1)(8,2)(6,4)	(9,1)(8,2)(6,4)	(9,1)(8,2)(6,4)
δ_u / δ_d 0.1	0.84 0.88 0.96	0.84 0.88 0.96	0.84 0.88 0.96
0.3	0.47 0.55 0.80	0.46 0.54 0.80	0.46 0.54 0.80
0.5	0.56 0.59 0.80	0.53 0.58 0.80	0.53 0.58 0.80
1.0	1.79 1.40 1.04	1.69 1.38 1.04	1.65 1.38 1.04
2.0	1.38 1.06 1.00	1.25 1.03 1.00	1.21 1.02 1.00

Note：in table 1,2,3,4, $\gamma = 3.0$, $n_0 = 5$, $(n_1, n_2) = (,)$. $\delta_d = 2.0$

Here is the average sample size used in all cases. Note that the loss coefficient r has no effect on the average sample size.

TABLE V. THE RATIO OF QUALITY LOSS ABOUT ASS $\bar{x}$ CHARTS AND THE MATCHED SSS $\bar{x}$ CHARTS DURING AN OUT-CONTROL PERIOD

$\bar{n}$	$p_U = 0.3, p_L = 0.7$			$p_U = 0.5, p_L = 0.5$			$p_U = 0.7, p_L = 0.3$			$p_U = 0.9, p_L = 0.1$		
	(9,1)	(8,2)	(6,4)	(9,1)	(8,2)	(6,4)	(9,1)	(8,2)	(6,4)	(9,1)	(8,2)	(6,4)
δ_u / δ_d 0.1	6.20	6.01	5.34	6.3 1	6.05	5.35	6.36	6.06	5.35	6.39	6.07	5.35
0.3	5.54	6.47	5.67	6.62	6.92	5.73	7.35	7.16	5.76	7.87	7.31	5.78
0.5	3.55	5.08	5.37	4.44	5.70	5.52	5.43	6.21	5.62	6.54	6.64	5.68
1.0	2.94	4.28	4.96	3.49	463	4.99	4.25	5.01	5.03	5.36	5.43	5.07
2.0	2.71	4.06	4.93	3.08	4.27	4.95	3.62	4.51	4.97	4.54	4.82	4.99

Note：in table 5, $\gamma = 3.0$, $n_0 = 5$, $(n_1, n_2) = (,)$. $\delta_d = 2.0$

TABLE VI. THE RATIO OF QUALITY LOSS ABOUT ASS $\bar{x}$ CHARTS AND THE MATCHED SSS $\bar{x}$ CHARTS DURING AN OUT-CONTROL PERIOD ($\delta_d = 2.0, \delta_U = 4.0$)

CP	($\delta_d = 2.0, \delta_U = 4.0$)			($\delta_U = 2.0, \delta_d = 4.0$)		
	$r = 0.5$	$r = 1$	$r = 5$	$r = 0.5$	$r = 1$	$r = 5$
	(8,2) (2,8)	(8,2)(2,8)	(8,2)(2,8)	(8,2) (2,8)	(8,2)(2,8)	(8,2)(2,8)
p_u 0.3	1.85 1.31	1.74 1.27	1.36 1.13	1.08 1.20	1.12 1.34	1.26 1.72
0.5	1.71 1.26	1.54 1.20	1.19 1.07	1.14 1.37	1.20 1.54	1.31 1.86
0.7	1.50 1.18	1.34 1.12	1.09 1.04	1.21 1.58	1.27 1.74	1.34 1.94