Journal of Management Science and Statistical Decision
Vol.7 No1.(2010) p105-109

The DiscreteTime Geo/Geo/1 Bernoulli Feedback Queue with Negative Customers and Multiple Working Vacations

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***Abstract*—The paper studies a discrete-time Bernoulli feedback queue with negative customers and multiple working vacations. The server works at a lower rate rather than completely stop service during the vacation period, after completing the service, the customer may leave the system with probability $\varepsilon\,(0 \le \varepsilon < 1)$, or feedback with probability $1-\varepsilon$ waiting for the next service. The rule of service is the first coming first served and the working vacation policy follows exhaustive service, negative customers only remove the positive customers one by one at the end and do not stay for some time. By using the matrix-geometric solutions, we obtain the equilibrium conditions of the system, then solve the steady-state distributions and stochastic decomposition of the queue length in the system.**

Keywords- discrete-time queue; negative customer; working vacation; feedback; matrix-geometric solution.

I. INTRODUCTION

Working vacation (WV) is firstly advanced by Servi and Finn[1],which is rooted in performance analysis of gateway router in fiber communication networks. Its characteristic is that the server works at a lower rate rather than completely stop service during the vacation period. In recent years, the queuing model with working vacations has been conducted in-depth study by many scholars. Such as [2].Along with the development of computer communications technology, we have greatly promoted the research and application of it. Take the web service for example, to keep the servers functioning well, virus scan is an important maintenance activity for the servers. It can be performed when the servers are idle. We look upon this type of work as a working vacation. It can also be performed when the servers are busy .We regard it as a regular service (the busy period). When virus scan is done, the servers will enter the idle state again and wait the use requests arrive. We can see it in literature [3].Using the queuing theory to solve the model above, we can easily apply mathematical method to optimize this model and make the solution more convenient.

In the communication system with external interference, when the receiver find the data was transmitted loss or error, the data will be required feedback to transfer again. Considering the conditions above, we introduce the negative customers and the Bernoulli feedback into the working vacations. The two Strategies are also studied by many people, such as [4-5].The we consider a discrete time queuing system with the Strategies above, which forms this paper. It makes the paper having more practical value.

II. MODEL DESCRIPTION AND EQUILIBRIUM CONDITION

For any real number $x \in [0,1]$, we let $\bar{x} = 1-x,\ n = 0,1,\cdots$.If we suppose n^- represent a moment before n and n^+ represent a moment after n, (n^-, n) and (n, n^+) are two very short time , both would be generally negligible. The model we studied here is as follows:

a) Two types of customers ,positive and negative , arrive separately and simultaneously at the end of slot (n^-, n), they arrive according to geometrical arrival process with probability p and q respectively.

b) The beginning and ending of service occur at slot division point $t = n$, the distribution of service time S_b in a regular busy period and S_v in a working vacation period are geometrically distributed $P\{S_b = k\} = \mu_b \cdot \bar{\mu}_b^{\,k-1}, P\{S_v = k\} = \mu_v \cdot \bar{\mu}_v^{\,k-1}, k \ge 1, 0 < \mu_b, \mu_v < 1.$

c) A server begins a working vacation at the epoch when the queue becomes empty, the distribution of vacation time V is $P\{V = k\} = \theta \cdot \bar{\theta}^{k-1},\quad k \ge 1,\quad 0 < \theta < 1.$

To make sure, we suppose the beginning and ending of vacation occur at the epoch (n, n^+). The server will take service at the rate of μ_b and when a service completes and there are no customers (means the positive customers, for the negative customers only takes the positive customers away without any delay) in system, the server will enter into a vacation. Customers arrive during the vacation and will be served at the rate of μ_{v} in the order of arriving at the system. When a working vacation ends, if there are no customers (the above) yet in the queue, another vacation is taken; Otherwise, the server switches service rate from μ_{v} to μ_{b}, and a regular busy period starts.

d) If the positive customers are not taken away when they are receiving service, then they will leave the system with probability $\varepsilon(0<\varepsilon\le 1)$ after the service, and feedback to the end of the queue with probability $1-\varepsilon$ for the next service

e) Aassume that inter-arrival times, service times and working vacation times are mutually independent. In addition,the service follows $FCFS$.

Based on the description above, the model we discussed is the late arrival system with immediate entrance (see in[6]).

Let L_n^+ be the number of customers in system at time n^+. According to above assumption, a customer who finishes service and leaves at $t=n$ does not reckon in L_n^+, let

$$J_n=\begin{cases}0, & \text{the system is on a working vacation period at time } n^+,\\ 1, & \text{the system is on a regular busy period at time } n^+.\end{cases}$$

Then $\{(L_n^+, J_n),\ \mathrm{n}\ge 0\}$ is a Markov chain with the state space

$$\Omega=\{(0,0)\}\cup\{(k,j):k\ge 1, j=0,1\}.$$

Using the lexicographical sequence for the states, the transition probability matrix can be written as the following block form:

$$\tilde{\mathrm{P}}=\begin{bmatrix} A_{00} & A_{01} & & & & & \\ A_{10} & A_1 & A_0 & & & & \\ A_{20} & A_2 & A_1 & A_0 & & & \\ & A_3 & A_2 & A_1 & A_0 & & \\ & & A_3 & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (1)$$

Where

$$A_{00}=\bar p+pq, A_{01}=\begin{pmatrix}\bar\theta p\bar q & \theta p\bar q\end{pmatrix}, A_{20}=\begin{pmatrix}\mu_v\bar p q\varepsilon & \mu_b\bar p q\varepsilon\end{pmatrix}^{\mathrm{T}},$$

$$A_{10}=\begin{pmatrix}\mu_v\overline{pq}\varepsilon+\mu_v pq\varepsilon+\bar p q & \mu_b\overline{pq}\varepsilon+\mu_b pq\varepsilon+\bar p q\end{pmatrix}^{\mathrm{T}},$$

$$A_0=\begin{pmatrix}\bar\theta(\bar\mu_v p\bar q+\mu_v p\bar q\bar\varepsilon) & \theta(\bar\mu_v p\bar q+\mu_v p\bar q\bar\varepsilon)\\ 0 & \bar\mu_b p\bar q+\mu_b p\bar q\bar\varepsilon\end{pmatrix},$$

$$A_3=\begin{pmatrix}\bar\theta\mu_v\bar p q\varepsilon & \theta\mu_v\bar p q\varepsilon\\ 0 & \mu_b\bar p q\varepsilon\end{pmatrix}, A_1=\begin{pmatrix}\bar\theta\delta_v & \theta\delta_v\\ 0 & \delta_b\end{pmatrix},$$

$$A_2=\begin{pmatrix}\bar\theta(\delta_v-\mu_v\bar p q\varepsilon)\delta & \theta(\delta_v-\mu_v\bar p q\varepsilon)\\ 0 & \eta_b-\mu_b\bar p q\varepsilon\end{pmatrix},$$

And

$$\delta_v \overset{\Delta}{=} \mu_v p\bar q\varepsilon+\mu_v\overline{pq}\,\bar\varepsilon+\bar\mu_v pq+\bar\mu_v\overline{pq}+\mu_v pq\bar\varepsilon$$

$$\delta_b \overset{\Delta}{=} \mu_b p\bar q\varepsilon+\mu_b\overline{pq}\,\bar\varepsilon+\bar\mu_b pq+\bar\mu_b\overline{pq}+\mu_b pq\bar\varepsilon$$

$$\eta_b \overset{\Delta}{=} \mu_b pq\varepsilon+\mu_b\overline{pq}\varepsilon+\bar p q, \eta_v \overset{\Delta}{=} \mu_v pq\varepsilon+\mu_v\overline{pq}\varepsilon+\bar p q,$$

$\beta \overset{\Delta}{=} \dfrac{\theta}{\bar\theta}$, $\alpha \overset{\Delta}{=} \dfrac{\bar\mu_b p\bar q+\mu_b p\bar q\bar\varepsilon}{\eta_b+\mu_b\bar p q\varepsilon}$. Based on the above, $\{(L_n^+, J_n),\ \mathrm{n}\ge 0\}$ is a quasi birth and death chain. We can regard (1) as the degenerative GI/M/1 type structure matrix (see in [7]). We can deal with it by using the matrix-geometric solutions. So the minimal non-negative solution R of matrix equation

$$R=A_0+RA_1+R^2A_2+R^3A_3 \quad (2)$$

is of important effect, the solution is called the rate matrix.

Lemma 1 If $\alpha<1$, the cubic equation

$$\mu_b\bar p q\varepsilon r^3+(\eta_b-\mu_b\bar p q\varepsilon)r^2+[(\delta_b+\mu_b p\bar q\bar\varepsilon)-1]r+(\bar\mu_b+\mu_b\bar\varepsilon)p\bar q=0$$

has three different real roots $r^{**}=1, r^*<-1, 0<r<1$.

Proof: we can easily verify $r=1$ is one root of the equation, suppose it as $(r-1)(ar^2+br+c)=0$, so we can get $b=\eta_b$, $a=\mu_b\bar p q\varepsilon, c=-(\bar\mu_b p\bar q+\mu_b p\bar q\bar\varepsilon)$, the discriminant Δ is more than 0 of $ar^2+br+c=0$. So the equation has two real roots, they are $r^*=(-\eta_b-\sqrt{\Delta})/(2\mu_b\bar p q\varepsilon), r=(-\eta_b+\sqrt{\Delta})/(2\mu_b\bar p q\varepsilon)$. Due to $\alpha<1$, $\Delta<(\eta_b+\mu_b\bar p q\varepsilon)^2$. So we can easily verify $r^*<-1, 0<r<1$.

Lemma 2 The cubic equation

$$\mu_v\bar p q\varepsilon x^3+(\eta_v-\mu_v\bar p q\varepsilon)x^2+[(\delta_v+\mu_v p\bar q\bar\varepsilon)-\frac{1}{\theta}]x+(\bar\mu_v+\mu_v\bar\varepsilon)p\bar q=0$$

has three different real roots $r_1^*<0, 0<r_1<1, r_1^{**}>1$.

Proof: the plot is figure 1, let the equation be $F(x)=0$, so $F(x)=\mu_v\bar p q\varepsilon(x^3+bx^2+cx+d)=\mu_v\bar p q\varepsilon f(x)$. where

$$b=\frac{\eta_v-\mu_v\bar p q\varepsilon}{\mu_v\bar p q\varepsilon}, c=\frac{\delta_v+\mu_v p\bar q\bar\varepsilon-1/\bar\theta}{\mu_v\bar p q\varepsilon}, d=\frac{\bar\mu_v p\bar q+\mu_v p\bar q\bar\varepsilon}{\mu_v\bar p q\varepsilon}.$$

Evidently $1/\bar\theta>1$, then $b>0, d>0, c<0$, so $f(x)=0$ is a real coefficient cubic equation and $f(1)<0, f(0)>0$, then there are at least one root denoted as r_1 and $0<r_1<1$. Because of $f'(x)=3x^2+2bx+c$, let $f''(x)=0$, we get $x=-b/3$, suppose $g=-b/3$, due to [7] and figure 1,we know $f(x)=0$ has only three real roots r_1^*, r_1, r_1^{**} and $r_1^*<0, 0<r_1<1. r_1^{**}>1$, evidently $F(x)=0$ is equivalent to $f(x)=0$,so the theorem is proved and the roots of equation can be seen in the appendix.

Theorem 1 If $\alpha < 1$, then the cubic equation (2) has the minimal non-negative solution

$$R = \begin{bmatrix} r_1 & r_2 \\ 0 & r \end{bmatrix} \quad (3)$$

where r_1 can be seen in appendix,

$$r_2 = \frac{\beta r_1}{(1-r_1)[\eta_b + \mu_b \bar{p} q \varepsilon (r_1 + r)]} \quad \text{and} \quad 0 < r_1 < 1.$$

Proof: A_0, A_1, A_2, A_3 are all upper triangular matrices, we suppose $R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$, substituting R and $A_i (i = 0,1,2,3)$ into equation (2), gives the following set of equations:

$$\begin{cases} r_{11} = \bar{\theta}[\mu_v \bar{p} q \varepsilon r_{11}^3 + (\eta_v - \mu_v \bar{p} q \varepsilon) r_{11}^2 + \delta_v r_{11} + (\bar{\mu}_v + \mu_v \bar{\varepsilon}) p \bar{q}] \quad (4.1) \\ r_{22} = \mu_b \bar{p} q \varepsilon r_{22}^3 + (\eta_b - \mu_b \bar{p} q \varepsilon) r_{22}^2 + \delta_b r_{22} + (\bar{\mu}_b + \mu_b \bar{\varepsilon}) p \bar{q} \quad (4.2) \\ r_{12} = \theta \mu_v \bar{p} q \varepsilon r_{11}^3 + \theta(\eta_v - \mu_v \bar{p} q \varepsilon) r_{11}^2 + \theta \delta_v r_{11} + \theta(\bar{\mu}_v + \mu_v \bar{\varepsilon}) p \bar{q} + \\ (\eta_b - \mu_b \bar{p} q \varepsilon)(r_{11} + r_{22}) r_{12} + \mu_b \bar{p} q \varepsilon (r_{11}^2 + r_{11} r_{22} + r_{22}^2) r_{12} + \delta_b r_{12} \quad (4.3) \end{cases}$$

To obtain the minimal non-negative solution of (2), taking $r_{22} = r$ in equation (4.2) (the others can be seen in lemma 1). In equation (4.1), we can obtain $r_{11} = r_1$ (The others can be seen in lemma 2 and $0 < r_1 < 1$). Substituting $r_{22} = r$ and $r_{11} = r_1$ into (4.3) together with (4.1) and (4.2) and simplifying it, we can get $r_{12}, r_{12} \overset{\Delta}{=} r_2$.

Theorem 2 The Markov Chain (MC) $\{(L_n^+, J_n), n \geq 0\}$ is positive recurrent if and only if $\alpha < 1$.

Proof : Based on Theorem 5.2.4 in $[6]$, MC is positive recurrent if and only if the spectral radius $SP(R) < 1$, and set of equations $(x_0, x_1, x_2) B[R] = (x_0, x_1, x_2)$ has positive solution, due to Theorem 1,we know R is reversible ,so

$$B[R] = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} + RA_{20} & A_1 + RA_2 + R^2 A_3 \end{bmatrix} =$$

$$\begin{bmatrix} \bar{p} + pq & \bar{\theta} p \bar{q} & \theta p \bar{q} \\ \eta_v + (\mu_v r_1 + \mu_b r_2) \bar{p} q \varepsilon & 1 - \dfrac{\bar{\theta}(\bar{\mu}_v + \mu_v \bar{\varepsilon}) p \bar{q}}{r_1} & \dfrac{(\bar{\mu}_b + \mu_b \bar{\varepsilon}) p \bar{q} r_2}{r r_1} - \dfrac{\theta(\bar{\mu}_v + \mu_v \bar{\varepsilon}) p \bar{q}}{r_1} \\ \eta_b + \mu_b \bar{p} q \varepsilon r & 0 & 1 - \dfrac{(\bar{\mu}_b + \mu_b \bar{\varepsilon}) p \bar{q}}{r} \end{bmatrix}$$

We can easily verify $B[R]$ is a stochastic matrix which ensures that set of equations with coefficient matrix $B[R]$ has positive solution. For example, let (x_0, x_1, x_2) be balance probability vector of $B[R]$. Note that $\alpha < 1 \Leftrightarrow r < 1$ and x_0, x_1, x_2 are all positive, so we have that $SP(R) = \max(r, r_1) < 1$ if and only if $\alpha < 1$

III. THE STEADY-STATE QUEUE SIZE AND ITS STOCHASTIC DECOMPOSITION

If $\alpha < 1$., let (L^+, J) be the stationary limit of (L_n^+, J_n), and its distribute function is denoted as

$$\pi_{kj}^+ = P\{L^+ = k, J = j\}, (k, j) \in \Omega, \pi_{00} = \pi_0, (\pi_{k0}^+, \pi_{k1}^+) = \pi_k.$$

Theorem 2 If $\alpha < 1$, the joint probability distribution of (L^+, J) is

$$\begin{cases} \pi_{00}^+ = K(\bar{\mu}_v + \mu_v \bar{\varepsilon}), \\ \pi_{k1}^+ = K r_2 \sum_{j=0}^{k-1} r_1^j r^{k-1-j}, \quad k \geq 1 \\ \pi_{k0}^+ = K r_1^k, \quad k \geq 1 \end{cases} \quad (5)$$

Where

$$K = \frac{(1-r_1)(1-r)}{r_2 + r_1(1-r) + (1-r)(1-r_1)(\bar{\mu}_v + \mu_v \bar{\varepsilon})}.$$

Proof: With the matrix-geometric solution method in theorem 5.2.4 (see $[6]$), we have $\pi_k = (\pi_{10}^+, \pi_{11}^+) R^{k-1}, k \geq 1,$ and $(\pi_{00}^+, \pi_{10}^+, \pi_{11}^+) B[R] = (\pi_{00}^+, \pi_{10}^+, \pi_{11}^+)$. Substituting $B[R]$ into the above relation, we obtain the set of equations

$$\begin{cases} \pi_{00}^+ = (\bar{p} + pq)\pi_{00}^+ + [\eta_v + (\mu_v r_1 + \mu_b r_2) \bar{p} q \varepsilon] \pi_{10}^+ + (\eta_b + \mu_b \bar{p} q \varepsilon r) \pi_{11}^+ \\ \pi_{10}^+ = \bar{\theta} p \bar{q} \pi_{00}^+ + [1 - \dfrac{\bar{\theta}(\bar{\mu}_v + \mu_v \bar{\varepsilon}) p \bar{q}}{r_1}] \pi_{10}^+ \\ \pi_{11}^+ = \theta p \bar{q} \pi_{00}^+ + [\dfrac{(\bar{\mu}_b + \mu_b \bar{\varepsilon}) p \bar{q}}{r r_1} r_2 - \dfrac{\theta(\bar{\mu}_v + \mu_v \bar{\varepsilon}) p \bar{q}}{r_1}] \pi_{10}^+ \\ \qquad + (1 - \dfrac{(\bar{\mu}_b + \mu_b \bar{\varepsilon}) p \bar{q}}{r}) \pi_{11}^+ \end{cases}$$

Let π_{00}^+ as a constant, we can use it to express π_{10}^+, π_{11}^+ .From (3), we can get the expression of R^k .Substituting R^{k-1} and $\pi_1 = (\pi_{10}^+, \pi_{11}^+)$ into π_k, we obtain

$$\pi_k = \frac{\pi_{00}^+}{\bar{\mu}_v + \mu_v \bar{\varepsilon}} \begin{pmatrix} r_1^k & r_2 \sum_{j=0}^{k-1} r_1^j r^{k-1-j} \end{pmatrix}, \quad k \geq 1.$$

Finally, constant factor π_{00}^+ can be determined by the normalization condition. $\pi_{00}^+ = K(\bar{\mu}_v + \mu_v \bar{\varepsilon})$. Hence, we get the joint distribution in (5).

From (5), we can obtain the state probabilities of a server in steady-state.

$$P(J=0) = K^*(1-r)(r_1 + (\bar{\mu}_v + \mu_v \bar{\varepsilon})(1-r_1)), P(J=1) = K^* r_2.$$

Where $K^* = [r_2 + r_1(1-r) + (1-r)(1-r_1)(\bar{\mu}_v + \mu_v \bar{\varepsilon})]^{-1}$.

The marginal distribution of the steady-state queue length L^+ at time n^+ in system is as follows

$$P\{L^+ = 0\} = \pi_{00}^+, \quad P\{L^+ = k\} = \pi_{k0}^+ + \pi_{k1}^+, \quad k \geq 1.$$

From (5) , we can get the probability generating function (PGF)of steady-state queue length:

$$L^{+}(z)=K^{*}(1-r_1)(1-r)[(\bar{\mu}_v+\mu_v\bar{\varepsilon})+\frac{r_1 z}{1-r_1 z}+\frac{r_2 z}{(1-r_1 z)(1-rz)}] \quad (6)$$

The average customers in system is

$$E(L^{+})=K^{*}[\frac{r_1(1-r)}{1-r_1}+\frac{r_2(1-r_1 r)}{(1-r_1)(1-r)}]$$

Now, we give the stochastic decomposition structure of the steady-state queue length in system:

Theorem 4 If $\alpha<1$ and $\mu_b>\mu_v$,the steady-state queue length L^{+} can be decomposed into the sum of two independent random variables: $L^{+}=L_0^{+}+L_d$, where L_0^{+} is the number of the steady-state customers in a corresponding classic Geo/Geo/1 queue without vacation, and follows a geometric distribution with parameter $1-r$; additional queue length L_d has the PGF:

$$L_d(z)=K^{*}[\delta_0+\delta_1 z+\delta_2 z(1-r_1)/(1-r_1 z)] \qquad (7)$$

Where

$$\delta_0=(\bar{\mu}_v+\mu_v\bar{\varepsilon})(1-r_1),\ \delta_1=(1-r_1)r\mu_v\varepsilon,\ \delta_2=(r_1+r_2-r).$$

Proof: With (6) , the PGF of the steady-state queue length L^{+} can be described as:

$$L^{+}(z)=(1-r)/(1-rz)\,L_d(z)=L_0^{+}(z)L_d(z),$$

Where $L_d(z)$ is same to (7). Note: In formula (7),

$$\delta_2=\frac{\varepsilon(\mu_b-\mu_v)(r_1+p/\bar{p})(r_1+q/\bar{q})}{\bar{p}q[\eta_b+\mu_b\bar{p}q\varepsilon(r_1+r)]}.$$

For $0<r_1<1,\ \mu_b>\mu_v,\eta_b>0,$ so $\delta_2>0$. Then, we can verify that $K^{*}=(\delta_0+\delta_1+\delta_2)^{-1}$, which prove that $L_d(z)$ is actually a PGF .

Theorem 4 indicates that L_d is a mixture of three random variables $\mu_b>\mu_v, L_d=K^{*}\delta_0X_0+K^{*}\delta_1X_1+K^{*}\delta_2X_2$, where $X_0=0, X_1=1, X_2$ follows a geometric distribution with parameter $1-r_1$ in the set of positive integer. Based on the above stochastic decomposition, we get means

$$E(L_d)=K^{*}[\delta_1+\delta_2/(1-r_1)],$$

$$E(L^{+})=r/(1-r)+K^{*}[\delta_1+\delta_2/(1-r_1)].$$

A special case: We can prove the Geo/ Geo/1/WV queue can be regarded as this model of which $q=0,\varepsilon=1$, so this model extend the corresponding results in discrete time Geo/ Geo/1 queue with multiple working vacations.

Appdendix

From [8] , $f(x)=0$ can become $y^3+Py+Q=0$, where $P=c-\frac{b^2}{3}, Q=\frac{2}{27}b^3-\frac{1}{3}bc+d$. Evidently $P<0, Q>0.$

We have known $f(x)$ has three different real roots, so $D<0$

We can solve $f(x)=0$ by formula $\begin{cases} R1=g+\sqrt[3]{A}+\sqrt[3]{B} \\ R2=g+\omega\sqrt[3]{A}+\omega^2\sqrt[3]{B} \\ R3=g+\omega^2\sqrt[3]{A}+\omega\sqrt[3]{B} \end{cases}$

where $A=-\frac{Q}{2}+\sqrt{\left(\frac{Q}{2}\right)^2+\left(\frac{P}{3}\right)^3}$, $B=-\frac{Q}{2}-\sqrt{\left(\frac{Q}{2}\right)^2+\left(\frac{P}{3}\right)^3}$,

the discriminant $D=\left(\frac{Q}{2}\right)^2+\left(\frac{P}{3}\right)^3$. Using *Matlab* , let $\mu_v=m, \theta=n, \varepsilon=x$ .we can give any group of values of p,q,m,n,x to calculate $R1, R2, R3$. Among them, there must be one is less than 0, one is more than 1 and the third is between (0, 1), so we can judge the value of r_1^{*}, r_1, r_1^{**} . Take $p=0.7, q=0.5, m=0.3, n=0.2, x=0.3$ for example, we obtain

$$b=13.4444, c=-56.5556, d=23.5926, P=-116.8066,$$

$$Q=457.0545, D=-6.8006e+003,$$

$$A=-2.2853e+002+8.2466e+001i,$$

$$B=-2.2853e+002-8.2466e+001i,$$

$$R1=2.9617, R2=-16.8781-0.0000i,$$

$$R3=0.4720.$$

Evidently, $r_1=R3,\ r_1^{*}=R2,\ r_1^{**}=R3.$

A. Figures

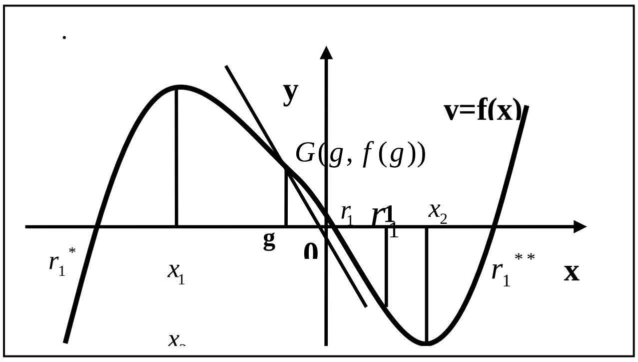

Figure 1.

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