Vol.7 No.4 (2010) p73-78 Journal of Management Science & Statistical Decision

IMPROVED PORTFOLIO OPTIMIZATION MODEL WITH TRANSACTION COST AND MINIMAL TRANSACTION LOTS

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ABSTRACT. *This paper first changes the binary objective model into single objective model by adopting linear weighed method. When studying the minimal transaction lots, this paper studies the integral constraint and the different minimal transaction lots. When studying the transaction cost, this paper studies the different transaction cost ratio. The paper then studies the situation with new investment and the soft constraint. Finally the paper establishes the portfolio optimization model with transaction cost and minimal transaction lot and conducts empirical analysis to the real data of the Shanghai stock market.*

Keywords: Portfolio; Transaction Cost; Minimal Transaction Lots

1. Introduction. Markowitz developed his theory of portfolio allocation under uncertainty in 1952. He put forward that the utilities of portfolio is the function of expected return rates and variance[3-4]. Usually high return rates are accompanied by high variance. When variance is fixed, investors pursue return rates as high as possible. When return rates are fixed, investors pursue variance as low as possible. Rational investors maximize their expected utilities by selecting effective portfolio. In Markowitz's mean-variance model, covariance must be calculated of risk assets and the calculation is very difficult. Sharp's capital asset price model divide risk into system risk and non-system risk. The model regards capital return as risk compensation. Sharp's single exponential model decreases the calculation work. Konno and Yamazaki[2] put forward the absolute variance can be utilized to measure risk and analyzed the Tokyo stock market.

Markowiz's classic mean-variance model neglects some important factors in the investment practice, such as the limitation of minimal transaction lots and transaction expense. In recent years, some portfolio optimization models put forward by Mansini and Grazia consider the minimal transaction lots[1,5,6], and the investment forming process is close to practice situation. The existed results indicate the mean-variance model with minimal transaction lots and transaction expense is a integral programming model with strict constraint. When the model only considers minimal transaction lots and does not consider transaction expense, the model is a NP-hard problem[1]. When consider minimal transaction lots, Mansini argued whether the solution can be found or not depended risk function, but he adopted the mean-absolute error model put forward by Konnoand Yamazaki. Because Mansini did not adopt Markowitz's mean-variance model, covariance between different assets were not considered and there existed estimation risk.

This paper consider different factors in the stock market, such as the limitation of investment value, the minimal transaction lots, transaction expense cost and share allotment. Finally this paper established a improved portfolio model with minimal transaction lots and transaction cost.

2. Markowiz's Mean-Variance Model. Markowitz's portfolio theories can be expressed as the following quadratic programming.

$$\min \quad \sigma^2 = \sum_{\mathrm{i}=1}^{\mathrm{n}} \sum_{j=1}^{n} x_{\mathrm{i}} x_j \sigma_{ij}$$

$$\text{s.t} \qquad \sum_{\mathrm{i}=1}^{\mathrm{n}} x_i r_i = r \tag{1}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} x_i = 1 \, ,$$

where n is the number of assets, x_i is the prooration ratio of i th asset, r_i is the expected return rate of i th asset, r is the expected return rate of portfolio, σ^2 is the variance of portfolio, and σ is the covariance of the return rate of i th asset and j th asset.

The double objective programming problem of minimizing the variance and maximizing the revenue can be transformed into the following parameters programming.

$$\max \quad (1-\lambda) \cdot \sum_{i=1}^{n} x_{\mathrm{i}} r_i - \lambda \cdot \sum_{\mathrm{i}=1}^{\mathrm{n}} \sum_{j=1}^{n} x_{\mathrm{i}} x_j \sigma_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \tag{2}$$

$$x_i \geq 0, i = 1, 2, ..., n$$

Where λ is risk aversion coefficient. The more λ is, the more risk aversion is. $0 \leq \lambda \leq 1$.

3. The Portfolio Model with Minimal Transaction Lots and Transaction Cost. In the practical transaction process, the minimal transaction lots usually exist. Let the minimal transaction lots C_i of i th stock can be expressed as

$$c_i = N_i p_i \tag{3}$$

$$N_i = \begin{cases} N_b, & \text{when } k_{i,t+1} \geq k_{i,t} \\ N_s, & \text{when } k_{i,t+1} < k_{i,t} \end{cases}$$

Where N_i is the minimal transaction lots of i th stock, N_b is the minimal number of i th stock for the buyer, N_s is the minimal number of i th stock for the seller, and p_i is the price of i th stock.

When consider the minimal transaction lots, the investment weights of the model are adjusted as

$$x_i = k_i c_i / I \, , \tag{4}$$

Where k_i is the unit of i th stock, I is the upper limitation of the investment.

When we adjust the portfolio, the transaction cost will occur. Take i th stock for example.

Let $x_{i,t}$ and $x_{i,t+1}$ be the investment proportion of t th period and $t+1$ th period. $TC_{i,t+1}$ is the transaction cost of i th stock of $t+1$ th period, so

$$TC_{i,t+1} = d_{i,t+1} \mid x_{i,t+1} - x_{i,t} \mid \tag{5}$$

Where $d_{i,t+1}$ is transaction cost rate of i th stock of $t+1$ th period. Assume that buying stock and selling stock have different transaction cost rates. Let m_b be transaction cost rate of buyer, and let m_s be transaction cost rate of seller, so

$$d_{i,t+i} = \begin{cases} m_b, & \text{when } k_{i,t+1} \geq k_{i,t} \\ m_s, & \text{when } k_{i,t+1} < k_{i,t} \end{cases} \tag{6}$$

The total transaction cost of the portfolio at $t+1$ th period can be expressed as

$$TC_{t+1} = \sum_{i=1}^{n} d_{i,t+1} \mid x_{i,t+1} - x_{i,t} \mid \tag{7}$$

Under the limitation condition of minimal transaction lots, when we adjust the portfolio, the total transaction cost of the i th stock can be expressed as

$$TC_{i,t+1} = d_{i,t+1} \mid k_{i,t+1} - k_{i,t} \mid \cdot c_{i,t+1} \tag{8}$$

Where $d_{i,t+1}$ is the transaction cost rate of i th stock of $t+1$ th period. $k_{i,t}$ and $k_{i,t+1}$ are the numbers of the investment respectively of t th period and $t+1$ th period. $c_{i,t+1}$ is the minimal transaction lot.

The total transaction cost of the portfolio can be expressed as

$$TC_{i,t+1} = \sum_{i=1}^{n} d_{i,t+1} \mid k_{i,t+1} - k_{i,t} \mid \cdot c_{i,t+1} \tag{9}$$

If we require the total investment is more than 95％ of the upper limit of the investment, so the portfolio optimization model with minimal transaction lots and transaction cost can be expressed as

$$\max \quad (1-\lambda) \cdot \sum_{i=1}^{n} x_{i,t+1} r_{i,t+1} - \lambda \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,t+1} x_{j,t+1} \sigma_{ij,t+1} - \frac{\sum_{i=1}^{n} d_{i,t+1} \cdot \mid k_{i,t+1} - k_{i,t} \mid \cdot c_{i,t+1}}{I_{t+1}}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{i,t+1} \leq 1$$

$$x_{i,t+1} = k_i c_i / I_{t+1}$$

$$c_i = N_i p_i$$

$$x_{i,t+1} \geq 0, i = 1, 2, \ldots, n$$

$$k_{i,t+1} \in N$$

$$0.95 I_{t+1} \leq \sum_{i=1}^{n} k_{i,t+1} c_{i,t+1} \leq I_{t+1}$$

$$N_i = \begin{cases} N_b, & \text{when } k_{i,t+1} \ge k_{i,t} \\ N_s, & \text{when } k_{i,t+1} < k_{i,t} \end{cases}$$

$$d_{i,t+i} = \begin{cases} m_b, & \text{when } k_{i,t+1} \ge k_{i,t} \\ m_s, & \text{when } k_{i,t+1} < k_{i,t} \end{cases}$$

Where m_b is the transaction cost rate of the buyer. m_s is the transaction cost rate of the seller. $k_{i,t}$ is the investment unit of i th stock of t th period. $k_{i,t+1}$ is the investment unit of i th stock of $t+1$ th period. $I_{i,t+1}$ is the upper limit of the investment of $t+1$ th period. $c_{i,t+1}$ is the minimal transaction lot of i th stock. N_i is the minimal transaction share of i th stock. p_i is the price of i th stock. $r_{i,t+1}$ is the expected return rate of of i th stock of $t+1$ th period. σ_{ij} is the covariance of return rate between i th stock and j th stock of t th period.

4. Positive Analysis. In our positive analysis, we select 8 stocks in Shanghai stock market. The stock codes are 600854, 600839, 600887, 600066, 600050, 600036, 600115 and 600702. Utilize Dazhihui software to download the history transaction data of every month from Jenuary, 2001 to March, 2007. The return rates are calculated according to the following formula.

$$r_t = \frac{(10 + a_t) \times P_t / 10 - P_{t-1} + D_t}{P_{t-1}} \tag{10}$$

Where r_t is the return rate of the stock in t th month. a_t indicates every 10 shares are allotted a_t shares in t th month. P_t is the close price of the last transaction day in t th month. P_{t-1} is the close price of the last transaction day in $t-1$ th month. D_t is the dividend in t th month.

The covariance matrix of return rates are shown in Table 1.

TABLE 1. The covariance matrix of return rates

0.0184	0.0102	0.0003	0.0038	0.0011	0.0015	0.0082	0.0071
0.0102	0.0165	0.0022	0.0038	0.0009	0.0021	0.0047	0.0061
0.0003	0.0022	0.0109	0.0039	0.0031	0.0037	0.0021	0.0033
0.0038	0.0038	0.0039	0.0166	0.0057	0.0034	0.0016	0.0066
0.0011	0.0009	0.0031	0.0057	0.0131	0.006	0.0043	0.0044
0.0015	0.0021	0.0037	0.0034	0.006	0.0088	0.0024	0.0027
0.0082	0.0047	0.0021	0.0016	0.0043	0.0024	0.0101	0.0059
0.0071	0.0061	0.0033	0.0066	0.0044	0.0027	0.0059	0.0081

Assume that the average return rate of the history months be the return rate in $t+1$ period, and the average close price of the history months be the close price in $t+1$ period. Let $k_{i,t} = 5, (i = 1, 2, ...8)$. Let $\lambda = 0.5$. Adopt the close price of the first transaction day in April, 2007. Assume that we buy 5 unit stock for every stock. Namely we buy 500 shares for every stock. Let the upper limit of the investment be RMB6000. The month return rates and prices of 8 stocks are shown in Table 2.

TABLE 2. The month return rates and prices of 8 stocks

Codes	600854	600839	600887	600066	600050	600036	600115	600702
Return rate	0.0191	0.0055	0.0259	0.0333	0.0166	0.0270	0.0076	0.0107
price	7.43	6.52	25.5	16.35	5.67	17.42	4.98	7.86

Let

$$N_i = \begin{cases} 100, & \text{when } k_{i,t+1} \geq 5 \\ 1, & \text{when } k_{i,t+1} < 5 \end{cases} \tag{11}$$

$$d_{i,t+i} = \begin{cases} 0.001, & \text{when } k_{i,t+1} \geq 5 \\ 0.002, & \text{when } k_{i,t+1} < 5 \end{cases} \tag{12}$$

Solve the programming with lingo10.0, and obtain different return rates and variance corresponding to different λ in Table 3.

TABLE 3. Return rates and variance corresponding to different λ

λ	(k1,k2,...,k8)	Return rates	variance
0	（0，0，5，36，0，0，0，0）	0.03272	0.01599
0.2	（0，0，5，36，0，0，0，0）	0.03272	0.01599
0.5	（5，0，5，19，0，9，0，0）	0.02981	0.00773
0.8	（6，0，6，7，0，12，13，0）	0.02460	0.00478
1	（2，5，5，3，0，10，26，14）	0.01728	0.00434

From Table 3 we can see that when the return rate increases, variance increases. Because we consider the minimal transaction lot, when $\lambda = 0$ and $\lambda = 0.2$, the return rates and variance are equal.

4. Conclusions. This paper adopted the linear weighed method to transform the double objective programming into single objective programming. Consider the minimal transaction lots and transction expense, this paper established the comprehensive non-linear programming. Finally this paper adopted the real data in the Shanghai stock market to make positive analysis.

Acknowledgment. This paper is supported by Humanities and Social Sciences Foundation of Ministry of Education of China (09YJC630129) and Philosophical and Social Science Foundation of Higher Education of Jiangsu Province of China (09SJD630059).

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